Lecture Notes
Fourier Analysis

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Contents

Notations:
If x is complex, $\Re(x)$ and $\Im(x)$ denote, respectively, the real and imaginary parts of x.
$\langle x, y\rangle$ denotes the inner product of x and y.

1 Background

Fourier analysis is important in modeling and solving partial differential equations related to boundary and initial value problems of mechanics, heat flow, electrostatics, and other fields.

1.1 Periodic functions

A function $f(x)$ is called periodic if $f(x)$ is defined for all real x, except possibly at some points, and if there is some positive number p, called a period of $f(x)$, such that

$$f(x+p) = f(x)$$

Remark 1. If p is a period of $f(x)$, then clearly $2p$, $3p, \ldots$ are also periods of $f(x)$.

The smallest positive period is called the fundamental period.

1.2 Orthogonal decomposition

In vector spaces equipped with inner products, we have the following essential theorem.

Theorem 2. *Suppose* $\{e_1, e_2, \ldots, e_n\}$ *is an orthogonal basis of a vector space* $\mathbf{V}$*. Then for every* $v \in \mathbf{V}$*, we have*

$$v = \frac{\langle v, e_1\rangle}{\langle e_1, e_1\rangle} e_1 + \cdots + \frac{\langle v, e_n\rangle}{\langle e_n, e_n\rangle} e_n$$

or, by using the norm notation,

$$v = \frac{\langle v, e_1\rangle}{||e_1||^2} e_1 + \cdots + \frac{\langle v, e_n\rangle}{||e_n||^2} e_n$$

□

Proof. Let $v \in \mathbf{V}$. Because $\{e_1, e_2, \ldots, e_n\}$ is a basis, there exists scalars $\alpha_1, \ldots, \alpha_n$ such that

$$v = \alpha_1 e_1 + \cdots + \alpha_n e_n$$

Taking the inner product with e_j $(j = 1, \ldots, n)$ yields

$$\langle v, e_j\rangle = \langle \alpha_1 e_1 + \cdots + \alpha_n e_n, e_j\rangle = \alpha_1 \langle e_1, e_j\rangle + \cdots + \alpha_n \langle e_n, e_j\rangle$$

But the basis is orthogonal, hence $\langle e_i, e_j\rangle = 0$ if $i \neq j$. This yields

$$\langle \alpha_1 e_1 + \cdots + \alpha_n e_n, e_j\rangle = \alpha_j \langle e_j, e_j\rangle$$

and thus

$$\langle v, e_j \rangle = \alpha_j \langle e_j, e_j \rangle \Longleftrightarrow \alpha_j = \frac{\langle v, e_j \rangle}{\langle e_j, e_j \rangle}$$

□

Recall that we have the following fact.

Fact (Inner product for functions, function spaces). *The set of all real-valued continuous functions $f(x)$, $g(x)$, ... $x \in [\alpha, \beta]$ is a real vector space under the usual addition of functions and multiplication by scalars. An inner product on this function space is*

$$\langle f, g \rangle = \int_{\alpha}^{\beta} f(x) g(x) \, \mathrm{d}x$$

and the norm of f is

$$||f(x)|| = \sqrt{\int_{\alpha}^{\beta} f(x)^2 \, \mathrm{d}x}$$

As a very useful special case, the trigonometric system is a function space with orthogonal basis $\{\sin nx, \cos nx, 1\}_{n=1}^{\infty}$. To check, we first notice that the functions are all periodic with period 2π. The inner product in this case is

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x) g(x) \, \mathrm{d}x$$

Noting that, by checking the area under the graphs and noting the shape of sine and cosine functions, we have

$$\begin{aligned} \int_{-\pi}^{\pi} \cos nx \mathrm{d}x = 0, \ \forall n = 1, 2, \ldots \\ \int_{-\pi}^{\pi} \sin nx \mathrm{d}x = 0, \ \forall n = 1, 2, \ldots \end{aligned} \tag{1}$$

This confirms the orthogonality between 1 and $\sin nx$ or $\cos nx$. Furthermore, using (1) we immediately know

$$\begin{aligned} \langle \cos nx, \cos mx \rangle = \int_{-\pi}^{\pi} \cos nx \cos mx \mathrm{d}x &= \int_{-\pi}^{\pi} \frac{\cos(n+m)x + \cos(n-m)x}{2} \mathrm{d}x \\ &= \frac{1}{2} \int_{-\pi}^{\pi} \cos(n+m)x \mathrm{d}x + \frac{1}{2} \int_{-\pi}^{\pi} \cos(n-m)x \mathrm{d}x \\ &= 0 + 0 \text{ as long as } n \neq m \end{aligned}$$

namely, $\cos nx$ and $\cos mx$ are orthogonal if $m \neq n$.

Similarly, as long as $n \neq m$ (by assumption, both m and n are positive integers), $\sin nx$ and $\sin mx$ are orthogonal. This is because

$$\langle \sin nx, \sin mx \rangle = \int_{-\pi}^{\pi} \sin nx \sin mx \mathrm{d}x = \int_{-\pi}^{\pi} \frac{\cos(n-m)x - \cos(n+m)x}{2} \mathrm{d}x = 0$$

Furthermore, regardless of the values of m and n, we have

$$\langle \sin nx, \cos mx \rangle = \int_{-\pi}^{\pi} \sin nx \cos mx \mathrm{d}x = \int_{-\pi}^{\pi} \frac{\sin(n+m)x + \sin(n-m)x}{2} \mathrm{d}x = 0$$

So $\sin nx$ and $\cos mx$ are always orthogonal.

Sketch the plots of sine and cosine functions. You should find the above results intuitive.

To perform the orthogonal decomposition under the trigonometric system, we need just one more thing–the norms of the basis functions. They are

$$\begin{aligned} ||\sin nx||^2 &= \langle \sin nx, \sin nx \rangle = \int_{-\pi}^{\pi} (\sin nx)^2 \mathrm{d}x = \int_{-\pi}^{\pi} \frac{1 - \cos 2nx}{2} \mathrm{d}x = \pi \\ ||\cos nx||^2 &= \int_{-\pi}^{\pi} (\cos nx)^2 \mathrm{d}x = \int_{-\pi}^{\pi} \frac{\cos 2nx + 1}{2} \mathrm{d}x = \pi \\ ||1||^2 &= \int_{-\pi}^{\pi} 1 \mathrm{d}x = 2\pi \end{aligned}$$

2 Fourier series

Fourier series is an extension of the orthogonal decomposition reviewed in the last section. Under the trigonometric system, the orthogonal decomposition of a function $f(x)$—provided that the decomposition exists (we will talk about the existence very soon)—is

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{2}$$

where

$$a_0 = \frac{\langle f(x), 1 \rangle}{\langle 1, 1 \rangle} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \mathrm{d}x \tag{3}$$

and

$$a_n = \frac{\langle f(x), \cos nx \rangle}{\langle \cos nx, \cos nx \rangle} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \mathrm{d}x, \; b_n = \frac{\langle f(x), \sin nx \rangle}{\langle \sin nx, \sin nx \rangle} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \mathrm{d}x \tag{4}$$

(2) is called the Fourier series of $f(x)$. (3) and (4) are called the corresponding Fourier coefficients.

Example 3. Find the Fourier coefficients of

$$f(x) = \begin{cases} -k & \text{if } -\pi < x < 0 \\ k & \text{if } 0 < x < \pi \end{cases}, \text{ and } f(x + 2\pi) = f(x)$$

Such functions occur as external forces acting on mechanical systems, electromotive forces in electric circuits, etc.

(Solution:

$$f(x) = \frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \dots\right)$$

)

Theorem 4 (Existance of Fourier Series). *Let $f(x)$ be periodic with period 2π and piecewise continuous in the interval $-\pi \le x \le \pi$. Furthermore, let $f(x)$ have a left-hand derivative and a right-hand derivative at each point of that interval. Then the Fourier series of $f(x)$ converges. Its sum is $f(x)$, except at points x_o where $f(x)$ is discontinuous. There the sum of the series is the average of the left- and right-hand limits of $f(x)$ at x_o.*

Hence a periodic function $f(x) = f(x + 2\pi)$ with

$$f(x) = 1/x,\ x \in [-\pi, \pi]$$

does not have a Fourier series extension, as it does not have a left- or right-hand derivative at $x = 0$.

A discontinuous periodic function, with period 2π and

$$f(x) = \begin{cases} 1, & -\pi < x < 0 \\ -1, & \pi > x \ge 0 \end{cases}$$

is piecewise continuous. At $x = 0$, it has a left-hand derivative of 0 and a right-hand derivative of 0. Hence the function has a Fourier series expansion.

2.1 Arbitrary period

The transition from period 2π to period $p = 2L$ can be done by a suitable change of scale. If $f(x)$ has a period $2L$, we let

$$v = \frac{x}{L}\pi$$

You can see, that $x = \pm L$ corresponds to $v = \pm\pi$. So

$$f(x) = f\left(\frac{L}{\pi}v\right)$$

is periodic in v with period 2π.

Now forget about $f(x)$. Focus just on the new $f\left(\frac{L}{\pi}v\right)$ with period 2π. The Fourier series is

$$f\left(\frac{L}{\pi}v\right) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos nv + b_n \sin nv\right)$$

where

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f\left(\frac{L}{\pi}v\right)\mathrm{d}v,\ a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{L}{\pi}v\right)\cos nv\mathrm{d}v,\ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{L}{\pi}v\right)\sin nv\mathrm{d}v \quad (5)$$

Changing back to the x notation, by using

$$f(x) = f\left(\frac{L}{\pi}v\right),\ v = \frac{x}{L}\pi,\ \mathrm{d}v = \frac{\pi}{L}\mathrm{d}x$$

we have

$$\boxed{f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x\right)} \tag{6}$$

with

$$\boxed{a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,\mathrm{d}x,\ a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}x\mathrm{d}x,\ b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}x\mathrm{d}x} \tag{7}$$

Example 5. Find the Fourier series of

$$f(x) = \begin{cases} 0 & \text{if } -2 < x < -1 \\ k & \text{if } -1 < x < 1 \\ 0 & \text{if } 1 < x < 2 \end{cases},\ p = 2L = 4,\ L = 2$$

Answer:

$$f(x) = \frac{k}{2} + \frac{2k}{\pi}\left(\cos\frac{\pi}{2}x - \frac{1}{3}\cos\frac{3\pi}{2}x + \frac{1}{5}\cos\frac{5\pi}{2}x - + \ldots\right)$$

□

2.2 Even and odd functions

It turns out that, if $f(x)$ is even or odd, the Fourier series can be significantly simplified. This is because, if $f(x)$ is an even function, then

$$\int_{-L}^{L} f(x)\sin nx\mathrm{d}x = 0$$

as the integral of any odd function is zero. Hence in (7) $b_n = 0$ so that (6) simplifies to

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{L}x$$

Furthermore, since $f(x)$ is even, we have

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,\mathrm{d}x = \frac{1}{L}\int_{0}^{L} f(x)\,\mathrm{d}x,\ a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}x\mathrm{d}x = \frac{2}{L}\int_{0}^{L} f(x)\cos\frac{n\pi}{L}x\mathrm{d}x$$

Similarly, if $f(x)$ is an odd function, then

$$\int_{-L}^{L} f(x)\cos nx\mathrm{d}x = 0$$

and

$$f(x)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi}{L}x,\ b_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\mathrm{d}x$$

The following Theorem is intuitive and very useful.

Theorem 6. *The Fourier series of a sum f_1+f_2 are the sums of the corresponding Fourier coefficients of f_1 and f_2. The Fourier coefficients of cf are c times the corresponding Fourier coefficients of f.*

Here is one example of using the results in this subsection.

Example 7 (Sawtooth wave)**.** Find the Fourier series of the function

$$f(x)=x+\pi \text{ if } -\pi<x<\pi \text{ and } f(x+2\pi)=f(x)$$

The basic idea is to decompose the function as

$$f(x)=f_1(x)+f_2(x),\ f_1(x)=x,\ f_2(x)=\pi$$

$f_1(x)$ is an odd function; $f_2(x)$ is an even function. Their Fourier coefficients are simpler to find.

2.3 Application example: ODE with special inputs

Consider a mass spring damper system

$$m\frac{\mathrm{d}^2}{\mathrm{d}t^2}y+b\frac{\mathrm{d}}{\mathrm{d}t}y+ky=r(t)$$

We have learned how to solve the nonhomogeneous ODE if $r(t)$ is a standard function such as sine, cosine, power functions. Difficulty arises if $r(t)$ is not a smooth function such as

$$r(t)=\begin{cases}t+\frac{\pi}{2} & \text{if } -\pi<t<0\\ -t+\frac{\pi}{2} & \text{if } 0<t<\pi\end{cases},\ r(t+2\pi)=r(t)$$

We can solve the problem by decomposing $r(t)$ as a Fourier series and then use linearity of the ODE to obtain the solution. The solution is actually very interesting. For the case of

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}y+0.05\frac{\mathrm{d}}{\mathrm{d}t}y+25y=r(t)$$

the solution looks like that in Fig. 1.

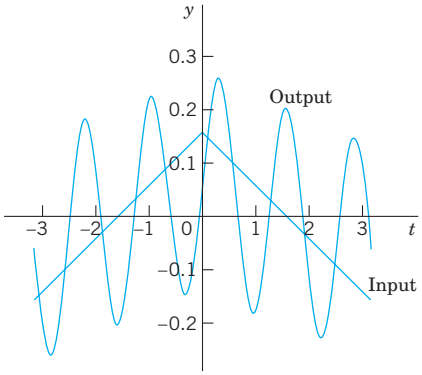

Figure 1: Fig. 277 from [EK]

3 Complex Fourier series

Instead of $\sin$ and $\cos$ functions, we can use $\{e^{jnx}\}_{j=-\infty}^{\infty}$ as the (complex) basis for Fourier series. For better physical intuitions, we usually write $n = \omega_s l$. The formula for Fourier series (when it exists) is

$$\boxed{f(x) = \sum_{l=-\infty}^{\infty} \frac{\left\langle f(x), e^{j\omega_s l x} \right\rangle}{\left\langle e^{j\omega_s l x}, e^{j\omega_s l x} \right\rangle} e^{j\omega_s l x}} \tag{8}$$

Fact (Inner product for complex functions). *The set of all complex-valued continuous functions $f(x)$, $g(x)$, ... $x \in [\alpha, \beta]$ is a complex vector space under the usual addition of functions and multiplication by scalars. An inner product on this function space is*

$$\langle g, f \rangle = \int_{\alpha}^{\beta} \overline{g(x)} f(x) \, \mathrm{d}x$$

and the norm of f is

$$||f(x)|| = \sqrt{\langle f, f \rangle} = \sqrt{\int_{\alpha}^{\beta} |f(x)|^2 \, \mathrm{d}x}$$

Fact 8. *$e^{j\omega_s l x}$ has a norm of $\sqrt{T_s} = \sqrt{2\pi/\omega_s}$*

Proof. By definition

$$||e^{j\omega_s l x}|| = \sqrt{\langle e^{j\omega_s l x}, e^{j\omega_s l x} \rangle} = \sqrt{\int_{-T_s/2}^{T_s/2} |e^{j\omega_s l x}|^2 \, \mathrm{d}x} = \sqrt{\int_{-T_s/2}^{T_s/2} \mathrm{d}x} = \sqrt{T_s}$$

□

The Fourier series expansion (8) hence simplifies to

$$\boxed{f(x) = \frac{1}{T_s} \sum_{l=-\infty}^{\infty} \left\langle f(x), e^{j\omega_s l x} \right\rangle e^{j\omega_s l x}}$$

Example 9 (Fourier series of an impulse train)**.** Show that

$$\sum_{l=-\infty}^{\infty}\delta\left(t-lT_s\right)=\frac{1}{T_s}\sum_{l=\infty}^{\infty}e^{j\omega_s lt},\ \omega_s=\frac{2\pi}{T_s}$$

where $\delta\left(x\right)$ is the Kronecker-delta function satisfying $\delta\left(x\right)=1$ if $x=0$ and $\delta\left(x\right)=0$ otherwise.

$\sum_{l=-\infty}^{\infty}\delta\left(t-lT_s\right)$ is periodic with period T_s. You can check that $e^{j\omega_s t}$ also has a period of T_s if $\omega_s=2\pi/T_s$.

For the Fourier coefficients, we have

$$\left\langle f\left(t\right),e^{j\omega_s lt}\right\rangle=\int_{-T_s/2}^{T_s/2}\delta\left(t\right)e^{j\omega_s lt}\mathrm{d}t=1$$

Hence

$$\sum_{l=-\infty}^{\infty}\delta\left(t-lT_s\right)=\frac{1}{T_s}\sum_{l=-\infty}^{\infty}\left\langle f\left(x\right),e^{j\omega_s lt}\right\rangle e^{j\omega_s lt}=\frac{1}{T_s}\sum_{l=\infty}^{\infty}e^{j\omega_s lt},\ \omega_s=\frac{2\pi}{T_s}$$

4 Fourier integral

What can be done to extend the method of Fourier series to nonperiodic functions? This is the idea of "Fourier integrals."

Let us consider an example of

$$f(x)=\begin{cases}1 & \text{if } -1<x<-1\\ 0 & \text{otherwise}\end{cases}\tag{9}$$

This is not a periodic function. Construct a rectangular wave

$$f_L(x)=\begin{cases}0 & \text{if } -L<x<-1\\ 1 & \text{if } -1<x<-1\\ 0 & \text{if } 1<x<L\end{cases},\ f_L\left(x+2L\right)=f_L(x)$$

We are going to make L increase from some small numbers to infinity, which recovers (9).

The function is even. The Fourier coefficients are

$$a_0=\frac{1}{2L}\int_{-1}^{1}\mathrm{d}x=\frac{1}{L},\ a_n=\frac{1}{L}\int_{-1}^{1}\cos\frac{n\pi x}{L}\mathrm{d}x=\frac{2}{L}\int_{0}^{1}\cos\frac{n\pi x}{L}\mathrm{d}x=\frac{2}{L}\frac{\sin\left(n\pi/L\right)}{n\pi/L}$$

As L increases, the frequency of $\sin\left(n\pi/L\right)$ increases.

More generally, consider any periodic function $f_L\left(x\right)$ of period $2L$ that can be represented by

$$f_L\left(x\right)=a_0+\sum_{n=1}^{\infty}\left[a_n\cos\omega_n x+b_n\sin\omega_n x\right],\ \omega_n=\frac{n\pi}{L}$$

where

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,\mathrm{d}x,\ a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}x\mathrm{d}x,\ b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}x\mathrm{d}x$$

or, in the notation of the newly introduced $\omega_n = n\pi/L$:

$$f_L(x) = \frac{1}{2L}\int_{-L}^{L} f_L(v)\,\mathrm{d}v + \frac{1}{L}\sum_{n=1}^{\infty}\left[\cos\omega_n x\int_{-L}^{L} f_L(v)\cos\omega_n v\mathrm{d}v + \sin\omega_n x\int_{-L}^{L} f_L(v)\cos\omega_n v\mathrm{d}v\right] \tag{10}$$

Notice that if we define

$$\Delta\omega = \omega_{n+1} - \omega_n = \frac{\pi}{L} \Rightarrow \frac{1}{L} = \frac{\Delta\omega}{\pi}$$

then (10) is actually

$$\begin{aligned} f_L(x) = {} & \frac{1}{2L}\int_{-L}^{L} f_L(v)\,\mathrm{d}v \\ & + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\Delta\omega\cos(\omega_n x)\int_{-L}^{L} f_L(v)\cos\omega_n v\mathrm{d}v + \Delta\omega\sin(\omega_n x)\int_{-L}^{L} f_L(v)\cos\omega_n v\mathrm{d}v\right] \end{aligned} \tag{11}$$

We want to have an understanding of the case when $L\to\infty$. Assume that $f(x) = \lim_{L\to\infty} f_L(x)$ is absolutely integrable, i.e. the following finite limit exists

$$\lim_{a\to-\infty}\int_a^0 |f(x)|\,\mathrm{d}x + \lim_{b\to\infty}\int_0^b |f(x)|\,\mathrm{d}x$$

Then

$$\lim_{L\to\infty}\frac{1}{2L}\int_{-L}^{L} f_L(v)\,\mathrm{d}v = 0$$

Let $g(\omega) = \cos(\omega x)\int_{-L}^{L} f_L(v)\cos\omega v\mathrm{d}v$. The first summation in (11) is

$$\sum_{n=1}^{\infty}\Delta\omega g(\omega_n) = g(\omega_1)\Delta\omega + g(\omega_2)\Delta\omega + \ldots$$

As $\Delta\omega = \pi/L\to 0$ if $L\to\infty$, the last term in (11) thus looks like an integration

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\left[\cos\omega x\int_{-\infty}^{\infty} f(v)\cos\omega v\mathrm{d}v + \sin\omega x\int_{-\infty}^{\infty} f(v)\cos\omega v\mathrm{d}v\right]\mathrm{d}\omega$$

Letting

$$\boxed{A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos\omega v\mathrm{d}v,\ B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin\omega v\mathrm{d}v}$$

yields

$$\boxed{f(x) = \int_0^{\infty}\left[A(\omega)\cos\omega x + B(\omega)\sin\omega x\right]\mathrm{d}\omega}$$

This is called a representation of $f(x)$ (no longer periodic) by a Fourier integral.

Remark 10. The above construction is only an intuition. Nonetheless, the conclusion is correct.

Theorem 11 (Existance of Fourier Integral). *If $f(x)$ is piecewise continuous in every finite interval and has a right hand derivative and a left-hand derivative at every point and if the integral*

$$\lim_{a\to-\infty}\int_a^0 |f(x)|\,\mathrm{d}x+\lim_{b\to\infty}\int_0^b |f(x)|\,\mathrm{d}x$$

exists, then $f(x)$ can be represented by a Fourier integral

$$\boxed{f(x)=\int_0^\infty [A(\omega)\cos\omega x+B(\omega)\sin\omega x]\,\mathrm{d}\omega} \tag{12}$$

with

$$\boxed{A(\omega)=\frac{1}{\pi}\int_{-\infty}^\infty f(v)\cos\omega v\mathrm{d}v,\ B(w)=\frac{1}{\pi}\int_{-\infty}^\infty f(v)\sin\omega v\mathrm{d}v} \tag{13}$$

At a point where $f(x)$ is discontinuous the value of the Fourier integral equals the average of the left- and right-hand limits of $f(x)$ at that point.

5 Fourier transform

With the Fourier integral formulas (12)-(13), we have

$$\begin{aligned}f(x)&=\int_0^\infty [A(\omega)\cos\omega x+B(\omega)\sin\omega x]\,\mathrm{d}\omega\\&=\frac{1}{\pi}\int_0^\infty\int_{-\infty}^\infty f(v)\left[\cos(\omega v)\cos(\omega x)+\sin(\omega v)\sin(\omega x)\right]\mathrm{d}v\mathrm{d}\omega\\&=\frac{1}{\pi}\int_0^\infty\left[\int_{-\infty}^\infty f(v)\cos(\omega x-\omega v)\,\mathrm{d}v\right]\mathrm{d}\omega\end{aligned}$$

Note that $\int_{-\infty}^\infty f(v)\cos(\omega x-\omega v)\,\mathrm{d}v$ is even in ω. Hence

$$\frac{1}{\pi}\int_0^\infty\left[\int_{-\infty}^\infty f(v)\cos(\omega x-\omega v)\,\mathrm{d}v\right]\mathrm{d}\omega=\frac{1}{2\pi}\int_{-\infty}^\infty\left[\int_{-\infty}^\infty f(v)\cos(\omega x-\omega v)\,\mathrm{d}v\right]\mathrm{d}\omega$$

To simplify the equations, we add an integral term that equals 0:

$$\frac{1}{2\pi}\int_{-\infty}^\infty\left[\int_{-\infty}^\infty f(v)\,i\sin(\omega x-\omega v)\,\mathrm{d}v\right]\mathrm{d}\omega=0,\ i=\sqrt{-1}$$

where the equality comes from the fact that $\int_{-\infty}^\infty f(v)\sin(\omega x-\omega v)\,\mathrm{d}v$ is an odd function of ω.

Adding up the last two equations gives

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^\infty\left[\int_{-\infty}^\infty f(v)\,e^{i\omega(x-v)}\mathrm{d}v\right]\mathrm{d}\omega=0$$

or, equivalently

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{i\omega x} \mathrm{d}\omega$$
$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(v) e^{-i\omega v} \mathrm{d}v = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} \mathrm{d}x$$

$F(\omega)$ is called the Fourier transform of $f(x)$; $f(x)$ is the inverse Fourier transform of $F(\omega)$. The above are often denoted as:

$$F(\omega) = \mathcal{F}(f(x)), \ f(x) = \mathcal{F}^{-1}(F(\omega))$$

Remark 12. Many people prefer to use a different normalization coefficient, and write:

$$\underline{f(x) = \int_{-\infty}^{\infty} F(\omega) e^{i\omega x} \mathrm{d}\omega}$$
$$\underline{F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} \mathrm{d}x}$$

Theorem 13 (Existance of Fourier transform). *If $f(x)$ is absolutely integrable on the x-axis and piecewise continuous on every finite interval, then the Fourier transform of $f(x)$ exists.*

Example 14. Find the Fourier transform of

$$f(x) = \begin{cases} 1, & |x| < 1 \\ 0, & \text{otherwise} \end{cases}$$

Solution: By definition

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-i\omega x} \mathrm{d}x = \frac{1}{-i\omega\sqrt{2\pi}} \left(e^{-i\omega} - e^{i\omega}\right) = \sqrt{\frac{\pi}{2}} \frac{\sin\omega}{\omega}$$

6 *Discrete Fourier analysis

Fourier series (FS) and Fourier transforms (FT) apply only to continuous-time functions. Discrete Fourier series (DFS) and discrete Fourier transform (DFT) are their discrete versions when the function $f(x)$ is defined at finitely many points. The main equations are summarized below. You can find the strong analogy between FS and DFS; as well as FT and DFT. For more details, see the reference [AO].

6.1 Discrete Fourier series

A periodic bounded discrete sequence $\tilde{x}[n]$ has a discrete Fourier series (DFS) expansion

$$\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] e^{j\frac{2\pi}{N}nk} \tag{14}$$

where the unnormalized DFS coefficients are

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j\frac{2\pi}{N}nk} \tag{15}$$

DFS properties Let $\tilde{x}[n]$ and $\tilde{X}[k]$ be defined as in (14) and (15). Then

Sequence	DFS coefficient
$\tilde{x}[n]$	$\tilde{X}[k]$
$\tilde{x}[n-m]$	$e^{-j\frac{2\pi}{N}km}\tilde{X}[k]$
$e^{j\frac{2\pi}{N}nl}\tilde{x}[n]$	$\tilde{X}[k-l]$
$\tilde{x}_3[n] = \sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m]$	$\tilde{X}_3[k] = \tilde{X}_1[k] \cdot \tilde{X}_2[k]$
$\tilde{X}[n]$	$N\tilde{x}[-k]$ (duality)

6.2 Discrete-time Fourier transform (DTFT)

A bounded infinite sequence $x[n]$ can be decomposed as

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\mathrm{d}\omega$$

where $X(e^{j\omega})$ is called the DTFT of $x[n]$ and is defined as

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}$$

Let $x^*[n]$ denote the complex conjugate sequence of $x[n]$. If $X(e^{j\omega})$ is the DFT of $x[n]$, then the following is true

Sequence	DTFT coefficient
$x^*[n]$	$X^*(e^{-j\omega})$
$x^*[-n]$	$X^*(e^{j\omega})$
$(x[n] + x^*[n])/2$	$\Re\{X(e^{j\omega})\}$
$(x[n] - x^*[n])/(2j)$	$\Im\{X(e^{j\omega})\}$
$e^{-j\omega n_0}$	$\delta[n-n_0]$
$e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$
$x[n] = a^n u[n]$	$X(e^{j\omega}) = \frac{1}{1-ae^{-j\omega}}$

6.3 Discrete Fourier transform (DFT)

A bounded sequence $x\,[n]$ defined on $n \in \{0, \ldots, N-1\}$ can be decomposed as

$$x[n] = \begin{cases} \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}kn} & , \quad n \in \{0, \ldots, N-1\} \\ 0 & , \quad n \notin \{0, \ldots, N-1\} \end{cases}$$

where $X[k]$ is the DFT of $x\,[n]$ and is defined as

$$X[k] = \begin{cases} \sum_{n=0}^{N-1} x[n] e^{-\frac{2\pi j}{N}kn} & , \quad k \in \{0, \ldots, N-1\} \\ 0 & , \quad k \notin \{0, \ldots, N-1\} \end{cases}$$

DFT properties

- the DFT pair $x[n] \longleftrightarrow X[k]$ (finite-length $x\,[n]$) is analogous to the DFS pair $\tilde{x}[n] \longleftrightarrow \tilde{X}[k]$ (infinite-length periodic $\tilde{x}\,[n]$)
- if $x[n]$ is real, then $X(k) = X^*[N-k]$
- if $N \geq L$, then DFT are samples of DTFT
- let $x_1\,[n] \textcircled{N} x_2\,[n] := \sum_{m=0}^{N-1} x_1[m] x_2[(n-m) \mod N]$. Then $x_1\,[n] \textcircled{N} x_2[n] \longleftrightarrow X_1[k] X_2[k]$
- if $x[n] = x[((-n))_N] = x[N-n]$, then $X[k]$ is real
- DFT coefficient table:

Sequence	DFT coefficient
$x[n]$	$X[k]$
$x^*[n]$	$X^*[((-k))_N]$
$x^*[((-n))_N]$	$X^*[k]$
$x[n]e^{j2\pi n\ell/N}$	$X[(k-\ell) \mod N]$
$x[((n-\ell))_N]$	$X[k]e^{-j2\pi k\ell/N}$
$\Re(x[n])$	$\frac{1}{2}\{X[((k))_N] + X^*[((-k))_N]\}$
$\Im(x[n])$	$\frac{1}{2}\{X[((k))_N] - X^*[((-k))_N]\}$
$\frac{1}{2}\{x[n] + x^*[((-n))_N]\}$	$\Re(X[k])$
$\frac{1}{2}\{x[n] - x^*[((-n))_N]\}$	$j\Im(X[k])$
$\frac{1}{2}\{x[n] + x[-n]\}$	$\sum_{n=0}^{N-1} x[n] \cos(\frac{2\pi kn}{2N-1})$
$x[n] \in \mathbb{R}$	$X[k] = X^*[N-k]$

Reference

[**EK**]: ERwin Kreyszig, Advanced Engineering Mathematics, 10th edition
[**AO**]: Alan Oppenheim et al., Discrete-time Signal Processing, 2nd edition