Estimating the Frequency Response of a Transfer Function

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1 Main results

Let u and y be the input and the output of a linear time-invariant (LTI) transfer function G_{yu}. The frequency response of G_{yu} is the quotient of Φ_{yu}, the cross power spectral density of u and y, and Φ_{uu}, the power spectral density of u:

$$G_{yu} = \frac{\Phi_{yu}}{\Phi_{uu}}$$

It is also true that

$$G_{yu} = \frac{\Phi_{yy}}{\Phi_{yu}}$$

2 The underlying theory

2.1 Definitions

- $X_{xx}(l)$: auto covariance of a stationary random process x, defined by

  $$X_{xx}(l) = E\left[(x(k) - E[x])(x(k+l) - E[x])\right]$$

  where $E[]$ is the operation of computing the mean.

- $X_{xy}(l)$: cross covariance between two stationary random processes x and y, defined by

  $$X_{xy}(l) = E\left[(x(k) - E[x])(y(k+l) - E[y])\right]$$

Under mild conditions (called ergodic) that are commonly satisfied in practice, auto and cross covariances can be computed by the ensemble averages, namely

$$\begin{aligned} X_{xx}(l) &= E\left[(x(k) - E[x])(x(k+l) - E[x])\right] = \overline{(x(k) - E[x])(x(k+l) - E[x])} \\ &= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{j=-N}^{N} (x(j) - E[x])(x(j+l) - E[x]) \end{aligned}$$

- $\Phi_{xx}(\omega)$: power spectral density is the Fourier transform of auto covariance, defined by

  $$\Phi_{xx}(\omega) = \sum_{l=-\infty}^{\infty} X_{xx}(l)\, e^{-j\omega l}$$

Remark: Given the time sequence of x and y, there are existing functions to calculate the power spectral densities in MATLAB.

2.2 Derivations

Consider passing a stationary random process $u(k)$ through an LTI transfer function $G(z)$. The resulting output is defined by the convolution:

$$y(k) = g(k) * u(k) = \sum_{i=-\infty}^{\infty} g(i)\, u(k-i)$$

where $g(k)$ is the impulse response of $G(z)$.

- if u is *zero mean* and ergodic, then

$$X_{uy}(l) = \overline{u(k) \sum_{i=-\infty}^{\infty} u(k+l-i)\, g(i)}$$

$$= \sum_{i=-\infty}^{\infty} \overline{u(k)\, u(k+l-i)} g(i) = \sum_{i=-\infty}^{\infty} X_{uu}(l-i)\, g(i) = g(l) * X_{uu}(l)$$

  similarly

$$X_{yy}(l) = \sum_{i=-\infty}^{\infty} X_{yu}(l-i)\, g(i) = g(l) * X_{yu}(l)$$

- in pictures we have

$$X_{uu}(l) \longrightarrow \boxed{G(z)} \longrightarrow X_{uy}(l)\,; \quad X_{yu}(l) \longrightarrow \boxed{G(z)} \longrightarrow X_{yy}(l)$$

- for a general LTI system

$$u(k) \longrightarrow \boxed{G(z) = \frac{b_n z^n + b_{n-1} z^{n-1} + \cdots + b_0}{z^n + a_{n-1} z^{n-1} + \cdots + a_0}} \longrightarrow y(k)$$

  convolution in time domain is multiplication in frequency domain:

$$Y(z) = G(z)\, U(z) \Leftrightarrow Y\left(e^{j\omega}\right) = G\left(e^{j\omega}\right) U\left(e^{j\omega}\right)$$

- hence for the auto/cross covariances:

$$X_{uu}(l) \longrightarrow \boxed{G(z)} \longrightarrow X_{uy}(l)\,; \quad X_{yu}(l) \longrightarrow \boxed{G(z)} \longrightarrow X_{yy}(l)$$

  we have

$$\boxed{\Phi_{yy}(\omega) = G\left(e^{j\omega}\right) \Phi_{yu}(\omega)}$$

$$\boxed{\Phi_{uy}(\omega) = G\left(e^{j\omega}\right) \Phi_{uu}(\omega)}$$