Notes

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Contents

1 Notations

We will be using the following notations throughout this set of notes:

Quantifiers

$\exists$ there exists

$\forall$ for all

: such that

Sets

$a \in A$ a is an element of the set A

Real and complex numbers

$\mathbb{R}$: the set of all real numbers

$\mathbb{C}$: the set of all complex numbers

So we can let $\mathcal{W}$ be the set of wolfs and $\mathcal{H}$ be the set of husky dogs, and write

$$\{h \in \mathcal{H} : \ \forall w \in \mathcal{W}, \ h \text{ is faster than } w\}$$

which defines us the set of super huskies that run faster than any wolfs.

Be careful. The writing of

$$\{h \in \mathcal{H} : \ \exists w \in \mathcal{W}, \ h \text{ is faster than } w\} \tag{1}$$

means different things. (1) describes the set of huskies that will run faster than at least one wolf.

Derivatives
If y is a function of x, then

$$\dot{y} = y^{'} = \frac{\mathrm{d}y}{\mathrm{d}x}$$
$$\ddot{y} = y^{''} = \frac{\mathrm{d}^2y}{\mathrm{d}x^2}$$

$\mathrm{d}y/\mathrm{d}x$ is read as "the derivative of y with respect to x", "$\mathrm{d}y$ by $\mathrm{d}x$", or "$\mathrm{d}y$ over $\mathrm{d}x$". The notation $y'(x)$ is read as y prime of x. This notation is attributed to Lagrange. The notation of $\dot{y}$ is attributed to Newton.[1] Often, the dot notation is used for functions of time. In this set of notes we often simplify and write y instead of $y\,(x)$, when the dependence is clear from the contents.

Partial derivatives
$\partial f/\partial x$, $\partial^2 f/\partial x^2$, and $\partial^2 f/\partial x \partial y$ are partial derivatives of f, if f is a function of x and y.

[1] Newton developed Newton's laws in 1686. He is an extremely brilliant scientist and in the meantime very eccentric. He was described as "...so absorbed in his studies that he forgot to eat".

2 Big picture: differential equations, applications

A differential equation can take the form of:

- an ordinary differential equation (ODE), which depends on a single variable. Important examples include

$$y^{'}(x) = ay(x) + b$$

  and

$$y^{''}(x) = a$$

- a partial differential equation (PDE), which depends on several variables. For instance, the two-dimensional Laplace equation

$$\frac{\partial^2 f(x,y)}{\partial x^2} + \frac{\partial^2 f(x,y)}{\partial y^2} = 0$$

  is a PDE.

Differential equations are important, as many physical concepts (e.g., linear/angular velocity and accelerations, currents, etc) are derivatives. Differential equations are central for modeling and analyzing their behaviors.

- Example: the main dynamics of many robot arms is modeled by a second-order differential equation.

3 Basics of ODE: order, general form, initial-value problem

- (Order) First-order ODEs, e.g.

$$y^{'} = 4x^3 y$$

  and

$$y^{'} - 4x^3 y = 0 \tag{2}$$

  contain only first-order derivatives

- General form of first-order ODEs:

$$y^{'} = f(x,y) \tag{3}$$

  The above is in the explicit form as $y^{'}$ is expressed directly as an *explicit* function of x and y. Correspondingly, (2) is in the *implicit* form.

- Initial-value problem: adding the initial value (initial condition) $y(x_0) = y_0$ to (3) gives

$$y^{'} = f(x,y),\ y(x_0) = y_0$$

– without specifying the initial condition, (3) itself can have infinitely many solutions. For instance, consider the free falling of a mass. We have

$$v^{'} = \frac{\mathrm{d}v}{\mathrm{d}t} = g \Rightarrow v\left(t\right) = gt + c$$

where the constant c can be an arbitrary number. However, if we constraint $v\left(0\right) = 0$, then it must be that

$$v\left(t\right) = gt$$

i.e., $c = 0$.

4 Separable ODEs

This section discusses solutions of ODEs that can be reduced to the form

$$g\left(y\right) y^{'} = f\left(x\right) \tag{4}$$

For instance, the ODE

$$y^{'} = \left(x+1\right) e^{-x} y^2$$

is equivalent to

$$y^{-2} y^{'} = \left(x+1\right) e^{-x} \text{ if } y \neq 0 \tag{5}$$

4.1 Solution of the standard problem

Solution of (4): (4) is equivalent to

$$g\left(y\right) \frac{\mathrm{d}y}{\mathrm{d}x} = f\left(x\right)$$

i.e.

$$g\left(y\right) \mathrm{d}y = f\left(x\right) \mathrm{d}x$$

So the variables x and y are separated. We can then integrate both sides, obtaining

$$\int g\left(y\right) \mathrm{d}y = \int f\left(x\right) \mathrm{d}x + c$$

where c is an arbitrary constant. Alternatively, if we know the initial condition $y\left(x_0\right) = y_0$, we can directly do

$$\int_{y_0}^{y(x)} g\left(\underline{y}\right) \mathrm{d}\underline{y} = \int_{x_0}^{x} f\left(\underline{x}\right) \mathrm{d}\underline{x} \tag{6}$$

Example 1. [Solution of (5)]: following the above general procedure, we obtain

$$\int y^{-2} \mathrm{d}y = \int \left(x+1\right) e^{-x} \mathrm{d}x + c_0$$

i.e.

$$\begin{aligned}
\int -\mathrm{d}y^{-1} &= -\int (x+1)\,\mathrm{d}e^{-x} + c_0\\
\iff -y^{-1} &= -(x+1)\,e^{-x} + \int e^{-x}\mathrm{d}x + c_0\\
\iff -y^{-1} &= -(x+1)\,e^{-x} - e^{-x} + c\\
\Rightarrow y &= \frac{1}{(x+2)\,e^{-x} - c}
\end{aligned}$$

If we additionally know the initial condition $y\left(x_0\right) = y_0$, then the value of c can be obtained from the constraint that

$$\begin{aligned}
y\left(x_0\right) = y_0 &= \frac{1}{\left(x_0+2\right)e^{-x_0} - c}\\
\Longrightarrow c &= \left(x_0+2\right)e^{-x_0} - y_0^{-1}
\end{aligned}$$

Alternatively, we can use (6) and get

$$\begin{aligned}
\int_{y_0}^{y(x)} \underline{y}^{-2}\mathrm{d}\underline{y} &= \int_{x_0}^{x} (\underline{x}+1)\,e^{-\underline{x}}\mathrm{d}\underline{x}\\
\Longrightarrow \int_{y_0}^{y(x)} -\mathrm{d}\underline{y}^{-1} &= -\int_{x_0}^{x} (\underline{x}+1)\,\mathrm{d}e^{-\underline{x}}\\
-y(x)^{-1} + y_0^{-1} &= -\left.(\underline{x}+1)\,e^{-\underline{x}}\right|_{x_0}^{x} + \int_{x_0}^{x} e^{-\underline{x}}\mathrm{d}\left(\underline{x}+1\right)\\
&= -\left[(x+1)\,e^{-x} - \left(x_0+1\right)e^{-x_0}\right] - e^{-x} + e^{-x_0}\\
&= -xe^{-x} - 2e^{-x} + x_0e^{-x_0} + 2e^{x_0}
\end{aligned}$$

and finally

$$y(x) = \frac{1}{(x+2)\,e^{-x} - \left(x_0+2\right)e^{-x_0} + y_0^{-1}}$$

You can check that the two approaches provide the same answer. We will not repeat the same steps in future discussions, and provide just one solution approach.

4.2 Extension: reduction to a separable form

Some non-separable ODEs can be reduced to a separable form. We discuss below systems in the form of

$$y' = f\left(\frac{y}{x}\right) \tag{7}$$

The trick is to introduce a new variable $u = y/x$, so that $f\left(y/x\right)$ can be simplified. More specifically, we have

$$y' = (ux)' = u'x + ux' = u'x + u$$

and hence (7) becomes

$$u^{'}x + u = f(u)$$

This ODE can be reduced to a separable form

$$\frac{u^{'}}{f(u) - u} = \frac{1}{x}$$
$$\Leftrightarrow \frac{\mathrm{d}u}{f(u) - u} = \frac{\mathrm{d}x}{x}$$

Example 2. Solve

$$2xyy^{'} = y^2 - x^2$$

Dividing both sides by $2xy$ yields

$$y^{'} = \frac{y}{2x} - \frac{x}{2y}$$

Letting $u = y/x$ and using $y^{'} = u^{'}x + u$, we get (after simplifications)

$$\frac{2u\mathrm{d}u}{u^2 + 1} = -\frac{\mathrm{d}x}{x}$$

After integration

$$\ln\left(1 + u^2\right) = -\ln|x| + c^* = \ln\left|\frac{1}{x}\right| + c^*$$
$$\Longrightarrow 1 + u^2 = \frac{c}{x}$$

Thus

$$1 + \frac{y^2}{x^2} = \frac{c}{x}$$

or

$$y^2 + x^2 = cx \Leftrightarrow \left(x - \frac{c}{2}\right)^2 + y^2 = \frac{c^2}{4}$$

□

5 Common differentiation and integration formulas

It is probably a good time to review formulas for common differentiation and integration.

Differentiation:

$$(u(x)\,v(x))^{'} = u^{'}(x)\,v(x) + u(x)\,v^{'}(x)$$
$$\left(\frac{u(x)}{v(x)}\right)^{'} = \frac{u^{'}(x)\,v(x) - u(x)\,v^{'}(x)}{v^2(x)}$$
$$\frac{\mathrm{d}u(x)}{\mathrm{d}x} = \frac{\mathrm{d}u(x)}{\mathrm{d}y(x)}\frac{\mathrm{d}y(x)}{\mathrm{d}x}$$

$$(x^n)' = nx^{n-1}$$
$$(e^{ax})' = ae^{ax}$$
$$(a^x)' = a^x \ln a$$

Integration:

$$\int uv' \mathrm{d}x = uv - \int u' v \mathrm{d}x$$
$$\int x^n \mathrm{d}x = \frac{x^{n+1}}{n+1} + c, \ n \neq -1$$
$$\int \frac{1}{x} \mathrm{d}x = \ln |x| + c$$
$$\int e^{ax} \mathrm{d}x = \frac{1}{a} e^{ax} + c$$
$$\int \sin x \mathrm{d}x = -\cos x + c$$

6 *Exact ODE

We know that

$$\mathrm{d}u(x,y)=\frac{\partial u}{\partial x}\mathrm{d}x+\frac{\partial u}{\partial y}\mathrm{d}y$$

So if

$$\mathrm{d}u(x,y)=0$$

it must be that

$$u(x,y)=c$$

This idea leads to a powerful solution method for a class of ODEs.

A first-order ODE in the form of

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0$$

is called an **exact differential equation** if $M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y$ is **exact** in the sense that there exists a function $u(x,y)$ such that

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=\mathrm{d}u(x,y)$$

Then we can get

$$u(x,y)=c$$

which is an implicit solution of the ODE. We will assume that $M(x,y)$ and $N(x,y)$ are continuous functions in the remaining discussions.

To check whether the ODE is an exact ODE, the following has to hold

$$M(x,y)=\frac{\partial u(x,y)}{\partial x},\ N(x,y)=\frac{\partial u(x,y)}{\partial y} \tag{8}$$

So it must be that

$$\frac{\partial M(x,y)}{\partial y}=\frac{\partial u(x,y)}{\partial y\partial x}=\frac{\partial u(x,y)}{\partial x\partial y}=\frac{\partial N(x,y)}{\partial x}$$

It turns out this condition of

$$\boxed{\frac{\partial M(x,y)}{\partial y}=\frac{\partial N(x,y)}{\partial x}} \tag{9}$$

is not only necessary but also sufficient for the ODE to be exact.

To solve the exact ODE, we look at (8) and notice that

$$u(x,y)=\int M(x,y)\,\mathrm{d}x+k(y) \tag{10}$$

where $k(y)$ is a function of y only. Be careful. $k(y)$ plays the role of a constant in the case of

$$p(x)=\frac{\mathrm{d}u(x)}{\mathrm{d}x}\Rightarrow u(x)=\int p(x)\,\mathrm{d}x+c$$

This is because $k(y)$ does not depend on x and hence

$$\frac{\partial k(y)}{\partial x} = 0$$

To determine $k(y)$, we use the fact that

$$\frac{\partial u(x,y)}{\partial y} = N(x,y)$$

in (8), and let

$$\frac{\partial}{\partial y}\left(\int M(x,y)\,\mathrm{d}x + k(y)\right) = N(x,y)$$

Example 3. Solve

$$\cos(x+y)\,\mathrm{d}x + \left(3y^2 + 2y + \cos(x+y)\right)\mathrm{d}y = 0$$

Solution:

Step 1: test exactness via checking (9).

Step 2: since

$$\frac{\partial u(x,y)}{\partial x} = M(x,y) = \cos(x+y)$$

thus

$$u(x,y) = \int \cos(x+y)\,\mathrm{d}x + k(y) = \sin(x+y) + k(y)$$

To find $k(y)$, we let

$$\frac{\partial u}{\partial y} = N(x,y) = \frac{\partial}{\partial y}\left(\sin(x+y) + k(y)\right) = \cos(x+y) + \frac{\mathrm{d}k(y)}{\mathrm{d}y}$$

and obtain

$$\frac{\mathrm{d}k(y)}{\mathrm{d}y} = N(x,y) - \cos(x+y) = 3y^2 + 2y$$

Therefore

$$k(y) = y^3 + y^2 + c^*$$

and

$$u(x,y) = \sin(x+y) + y^3 + y^2 = c$$

We have now an implicit solution $\sin(x+y) + y^3 + y^2 = c$.

Step 3: verify that

$$\mathrm{d}u = \frac{\partial u}{\partial x}\mathrm{d}x + \frac{\partial u}{\partial y}\mathrm{d}y = 0$$

which is indeed true.

If $M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0$ is not in an exact form, frequently we can try to find an integration factor to achieve the exactness. This is done by multiplying an function F on both sides to get

$$FM(x,y)\,\mathrm{d}x + FN(x,y)\,\mathrm{d}y = 0$$

such that the new differential equation is exact. Commonly it is sufficient to let F be a function of x only. You can work out the details following this big picture.

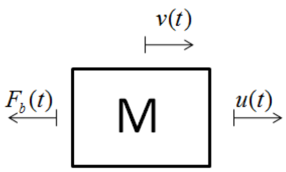
7 First-order linear ODE

A first-order ODE is linear if it can be transformed into the form of

$$\boxed{y^{'} + p(x)\, y = r(x)} \tag{11}$$

Here $r(x)$ is sometimes called the input to the ODE and y is called the output of the ODE. The ODE is linear with respect to y and $y^{'}$ and the input $r(x)$.

Example 4. Let a horizontal force u be applied to a mass M on a flat plane, forcing the mass to move at the velocity of v. There is wind resistance in the form of $F_b(t) = bv$.



Then Newton's Laws give

$$Mv^{'} = u - bv$$

i.e.

$$v^{'} + \frac{b}{M} v = \frac{u}{M} \tag{12}$$

which is a first-order ODE.

As another example

$$y^{'} \cos x + y \sin x = x$$

is also a first-order ODE, as it can be transformed to the standard form

$$y^{'} + y \tan x = x \sec x$$

To understand the solution concepts, we first look at a special case in the next subsection.

7.1 Homogeneous linear ODE

When $r(x) = 0$, (11) is

$$y^{'} + p(x)\, y = 0 \tag{13}$$

This is called a homogeneous linear first-order ODE. This special case is not difficult to solve. It is equivalent to

$$\frac{\mathrm{d}y}{y} + p(x)\, \mathrm{d}x = 0 \tag{14}$$

We will usually be focusing on the case of $y > 0$, then

$$\int \mathrm{d} \ln y + \int p(x)\, \mathrm{d}x = 0$$

$$\implies \ln y = -\int p(x)\, \mathrm{d}x + c^*$$

$$\implies y \boxed{= ce^{-\int p(x) \mathrm{d}x}},\ c = e^{c^*} \tag{15}$$

The above solution is called the homogeneous solution of (11). This formula will be repeatedly used in future discussions.

Example 5. Recall the moving mass ODE (12). If the applied force is zero, i.e. $u(t) = 0$, we can apply the solution technique in this section and get

$$v(t) = ce^{-\frac{b}{M}t}$$

This is an exponentially decaying function. If we know the initial velocity $v(0)$, we can determine the constant c from $v(0) = ce^0 = c$. So the magnitude of the velocity decreases in the presence of the resistance force.

Exercise 6. For (14), if we allow both $y > 0$ and $y \leq 0$, show that the full homogeneous solution is

$$y = ce^{-\int p(x)\mathrm{d}x}, \ c = \pm e^{c^*}$$

7.2 Nonhomogeneous linear ODE

Moving on for the **general case** of (11), we take a closer look at the structure of $y' + p(x)y$. Multiplying some $F(x)$ on both sides of (11) yields

$$F(x)y' + F(x)p(x)y = F(x)r(x) \tag{16}$$

The left hand side of the equation looks like a derivative:

$$\frac{\mathrm{d}F(x)y(x)}{\mathrm{d}x} = F(x)y'(x) + F'(x)y(x) \tag{17}$$

If this is true, then (16) is nothing but

$$\frac{\mathrm{d}F(x)y(x)}{\mathrm{d}x} = r(x)F(x)$$

So we can integrate to get

$$F(x)y(x) = \int r(x)F(x)\mathrm{d}x + c$$

$$y(x) = \frac{\int r(x)F(x)\mathrm{d}x + c}{F(x)} \tag{18}$$

The only remaining thing is to find out what $F(x)$ is. Matching (17) with (16), we get

$$F'(x) = F(x)p(x) \tag{19}$$

namely

$$\frac{\mathrm{d}F(x)}{F(x)} = p(x)\mathrm{d}x$$

$$\Longrightarrow F(x) = c^* e^{\int p(x)\mathrm{d}x}$$

Or we can use directly the formula of the homogeneous solution in the last section, by noticing that (19) is nothing but $F^{'}(x) - F(x)p(x) = 0$ [recall the form of the homogeneous ODE (13)].

Any non-zero constant c^* would work here. Let us pick $c^* = 1$ without loss of generality and define

$$F(x) = e^h, \; h = \int p(x)\,\mathrm{d}x$$

Combining the result with (18) gives the full solution

$$\boxed{y(x) = e^{-h}\int r(x)\,e^h \mathrm{d}x + ce^{-h}, \; h = \int p(x)\,\mathrm{d}x} \tag{20}$$

Exercise 7. Choose a different c^* and verify that the solution is still in the form of (20). Explain why this is true for a general c^*.

Observations: looking at (20) and (15), we see that the general solution of the ODE is

$$\boxed{\text{full solution} = \text{response due to } r(x) + \text{the homogeneous solution}}$$

This fact is very important and will be revisited later in this set of notes.

Example 8. For the moving mass ODE, we have

$$v^{'}(t) + \frac{b}{M}v(t) = \frac{u(t)}{M}$$

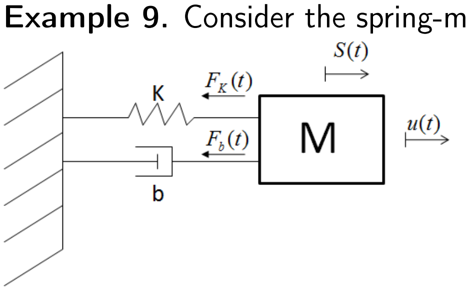
This is a particular case where $p(t)$ is constant. We have $h = \frac{b}{M}t$ and

$$v(t) = e^{-\frac{b}{M}t}\int \frac{u(t)}{M}e^{\frac{b}{M}t}\mathrm{d}t + ce^{-\frac{b}{M}t}$$

8 Second-order linear ODE

Motivation: second-order ODEs are extremely important in engineering applications. For instance, the principle dynamics of a rotating motor can be modeled by a second-order linear ODE. As another example, mechanical and electrical vibrations are essentially governed by second-order ODEs.

Example 9. Consider the spring-mass-damper system:



The position of the mass $S(t)$ is measured relative to the un-stretched position of the spring. The assumptions are

- an external force $u(t)$ is applied to the mass.
- the spring element has mass zero and it generates a braking force $F_K(t)$ proportional to the mass position $S(t)$:

  $$F_K(t) = KS(t)$$

  where K is the spring constant.
- the position of the mass is also affected by a viscous damper, which supplies a braking force $F_b = bS^{'}$ proportional to, and opposing the velocity of the mass. b is the friction coefficient.

Using Newton's law's we can easily write the equations of motion as

$$MS^{''}(t) = -KS(t) - bS^{'}(t) + u(t)$$

The **general form** of second-order linear ODEs is

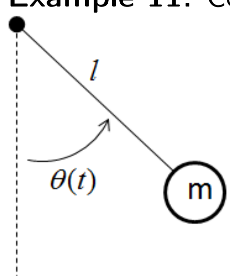
$$\boxed{y^{''} + p\left(x\right)y^{'} + q\left(x\right)y = r\left(x\right)} \tag{21}$$

$p\left(x\right)$ and $q\left(x\right)$ are called coefficients of the ODE. The left hand side is linear in y and its derivatives.

Example 10. $y^{''} + y = 1$ is a 2nd-order linear ODE. $y^{''}y - xy^{'} = 0$ however is not a linear ODE.

□

Example 11. Consider the pendulum shown below.



Let the pendulum has mass m Kg which is concentrated at the end point and length l meters. The angle θ is measured, and a torque $u(t) = T_c(t)$ is applied to the pivot point. The moment of inertia about the pivot point is $I = ml^2$. By analyzing the rigid body dynamics and writing Euler's equation for the pendulum, we can readily arrive at a differential equation model for this system:

$$I\theta(t)^{''} = T_c - mglsin(\theta(t))$$

namely

$$\theta(t)^{''} = \frac{1}{ml^2}T_c - \frac{g}{l}sin(\theta(t))$$

which is a second order nonlinear differential equation, and can not be transformed to the general form (21).

Similar to first-order ODEs, to solve (21), we first consider the homogeneous solution.

8.1 Homogeneous solution

This is again for the special case where $r(x) = 0$ and

$$y'' + p(x)y' + q(x)y = 0 \tag{22}$$

Let us take a look at the solution properties before digging into the actual calculation.

Theorem 12 (Fundamental theorem of homogeneous linear ODEs, second-order case). *If y_1 and y_2 are homogeneous solutions of $y'' + p(x)y' + q(x)y = 0$, then $y = c_1y_1 + c_2y_2$ is also a homogeneous solution for (22).*

The proof is not difficult. As

$$y_1'' + p(x)y_1' + q(x)y_1 = 0$$
$$y_2'' + p(x)y_2' + q(x)y_2 = 0$$

so

$$c_1y_1'' + p(x)c_1y_1' + q(x)c_1y_1 = 0$$
$$c_2y_2'' + p(x)c_2y_2' + q(x)c_2y_2 = 0$$

Adding the two equations up gives

$$(c_1y_1 + c_2y_2)'' + p(x)(c_1y_1 + c_2y_2)' + q(x)(c_1y_1 + c_2y_2) = 0$$

i.e. $c_1y_1 + c_2y_2$ satisfies (22).

In many cases the ODE coefficients are constant: $p(x) = a$, $q(x) = b$. We focus first on solving the following homogeneous 2nd-order linear ODEs with constant coefficients:

$$y'' + ay' + by = 0 \tag{23}$$

Take $y = e^{\lambda x}$, then $y' = \lambda e^{\lambda x}$ and $y'' = \lambda^2 e^{\lambda x}$. (23) becomes

$$\left(\lambda^2 + a\lambda + b\right)e^{\lambda x} = 0$$

As $e^{\lambda x} \neq 0$ for any x, we can divide both sides by $e^{\lambda x}$ and get

$$\boxed{\lambda^2 + a\lambda + b = 0}$$

This is the **characteristic equation** of the ODE, the solutions of which are

$$\lambda_1 = \frac{-a + \sqrt{a^2 - 4b}}{2}, \ \lambda_2 = \frac{-a - \sqrt{a^2 - 4b}}{2} \tag{24}$$

So $y_1 = e^{\lambda_1 x}$ and $y_2 = e^{\lambda_2 x}$ are solutions of (23). Using now Theorem 12, we know $y = c_1e^{\lambda_1 x} + c_2e^{\lambda_2 x}$ is a general homogeneous solution.

With the big picture drawn, let us now be more careful about the details. Observe in (24), that depending on the values of $a^2 - 4b$, the solutions (roots) of the characteristic equation take three different cases.

- Case I: $a^2 - 4b > 0$. Then λ_1, $\lambda_2 \in \mathbb{R}$ (two distinct real roots), and y_1, y_2 are real exponential functions.

- Case II: $a^2 - ab < 0$. Then λ_1 and λ_2 are complex numbers. Let $\sqrt{a^2 - 4b} = i\omega$ and

  $$\lambda_1 = -\frac{a}{2} + i\omega$$
  $$\lambda_2 = -\frac{a}{2} - i\omega$$

  then

  $$\begin{aligned} y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ &= c_1 e^{-\frac{1}{2}ax} e^{i\omega x} + c_2 e^{-\frac{1}{2}ax} e^{-i\omega x} \end{aligned}$$

  Using the Euler equation

  $$e^{i\omega x} = \cos \omega x + i \sin \omega x$$

  we can further simplify the result to

  $$\begin{aligned} y &= c_1 e^{-\frac{1}{2}ax} \left[\cos \omega x + i \sin \omega x\right] + c_2 e^{-\frac{1}{2}ax} \left[\cos \omega x - i \sin \omega x\right] \\ &\triangleq e^{-\frac{1}{2}ax} \left(A \cos \omega x + B \sin \omega x\right) \end{aligned}$$

  Often we are more interested in obtaining real instead complex solutions, and select A, B from the set of real numbers. In other words, if we have a complex root $\lambda = \gamma + i\omega$, we commonly write

  $$y = c_1 e^{\gamma x} \cos \omega x + c_2 e^{\gamma x} \sin \omega x$$

- Case III: $a^2 - 4b = 0$. Then $\lambda_1 = \lambda_2 = -\frac{a}{2} \in \mathbb{R}$ (real double root). $y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} = (c_1 + c_2) e^{-ax/2}$ in this case does not provide more information than $y_1 = e^{\lambda_1 x}$ or $y_2 = e^{\lambda_2 x}$. I will state the correct construction first and explain the reason afterwords.
  Do instead $y_1 = e^{-ax/2}$ and $y_2 = xe^{-ax/2}$. Then let

  $$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} = (c_1 + c_2 x) e^{-ax/2}$$

  The new y_1 and y_2 are linearly independent, and actually form a basis for the general solution. We will discuss more about this later.

The obtaining of $y_2 = xe^{-ax/2}$ in Case III comes from a technique called reduction of order, which we will learn next.

Find a second solution if one solution is known–reduction of order: In

$$y'' + p(x)\,y' + q(x)\,y = 0$$

if we know one solution y_1 defined on an open interval I and want to find a linearly independent solution y_2 on I, we substitute

$$y = y_2 = uy_1,\ y' = u'y_1 + uy_1',\ y'' = u''y_1 + 2u'y_1' + uy_1''$$

into the ODE and get

$$u''y_1 + 2u'y_1' + uy_1'' + p\left(u'y_1 + uy_1'\right) + quy_1 = 0$$

which is equivalent to

$$u''y_1 + u'\left(2y_1' + py_1\right) + u\left(y_1'' + py_1' + qy_1\right) = 0$$

Now, since y_1 is a solution, $y_1'' + py_1' + qy_1$ has to equal 0, namely the above equation is simplified to

$$u''y_1 + u'\left(2y_1' + py_1\right) = 0$$

Define new functions $u' = U$ and $u'' = U'$. We have

$$U'y_1 + U\left(2y_1' + py_1\right) = 0$$

which is a first-order ODE that we have studied how to solve. The solution is

$$U = c^*\frac{1}{y_1^2}e^{-\int p(x)\mathrm{d}x}$$

and thus (take $c^* = 1$)

$$\boxed{y_2 = uy_1 = y_1\int U\mathrm{d}x,\ U = \frac{1}{y_1^2}e^{-\int p(x)\mathrm{d}x}}$$

In Case III, we have $y_1 = e^{-ax/2}$ and $p(x) = a$, so

$$U = e^{ax}e^{-\int a\mathrm{d}x} = \text{constant} = c$$

Take $c = 1$ and we can get

$$y_2 = e^{-ax/2}x$$

Example 13. Solve the following ODEs:

i,

$$y'' + y' - 2y = 0,\ y(0) = 4,\ y'(0) = -5$$

(Answer: $y\left(x\right)=3e^{-2x}+e^{x}$)

ii,

$$y^{''}+y^{'}+0.25y=0,\ y\left(0\right)=3,\ y^{'}\left(0\right)=-3.5$$

(Answer: $y\left(x\right)=3e^{-0.5x}-2xe^{-0.5x}$)

iii,

$$y^{''}+0.4y^{'}+9.04y=0$$

(Answer: $y\left(x\right)=e^{-0.2x}\left(A\cos 3x+B\sin 3x\right)$)

8.2 Existence and uniqueness of solution

In Theorem 12, we have directly assumed that y_1 and y_2 are solutions of

$$y^{''} + p(x)\, y^{'} + q(x)\, y = 0 \tag{25}$$

One natural question is that whether these solutions actually exist and are unique. For the constant-coefficient ODE where $p(x) = a$ and $q(x) = b$, no special attention needs to be paid as we have explicitly calculated all the three possible Cases. It turns out that under mild conditions, (25) always has solutions. Furthermore, if we add initial conditions to (25), the solution will not only exist but also be unique.

Theorem 14 (The Fundamental Theorem of Ordinary Differential Equations). *If $p(x)$ and $q(x)$ are continuous functions on some open interval I, then the initial-value problem*

$$y^{''} + p(x)\, y^{'} + q(x)\, y = 0,\ y(x_0) = y_0,\ y^{'}(x_0) = y_0^{'}$$

has one unique *solution on the interval I.*

□

Theorem 15 (Existance of a general solution). *If $p(x)$ and $q(x)$ are continuous on an open interval I, then the ODE*

$$y^{''} + p(x)\, y^{'} + q(x)\, y = 0$$

has a general solution on I.

□

The above two theorems explain the existence of general solutions to the ODE and the uniqueness of solutions to the initial-value problem. The next theorem tells that any solution to (25) can be expressed in the form of $c_1 y_1(x) + c_2 y_2(x)$, provided that $y_1(x)$ and $y_2(x)$ are wisely chosen.

Theorem 16 (A general solution includes all solutions). *If (25) has continuous coefficients $p(x)$ and $q(x)$ on some open interval I, then any solution of (25) on I is of the form*

$$Y(x) = C_1 y_1(x) + C_2 y_2(x)$$

where $y_1(x)$ and $y_2(x)$ form a basis of solutions of (25).

You can find some intuition of Theorem 16 from Theorem 12, which actually gives useful ideas about the solution structure of the ODE. Think about the coordinates of a point in an xy plane: for any point (x, y), we can always write the point as

$$(x, y) = x(1, 0) + y(0, 1)$$

where $(1, 0)$ and $(0, 1)$ form the "basis" of the xy plane. In Theorem 16 and Theorem 12, $y = c_1 y_1 + c_2 y_2$ is a (linear) combination of y_1 and y_2. If y_1 and y_2 are very different in structure, in the sense that they express the solution in two different perspectives and there is no way that y_1 can be "obtained" from y_2 (analogy to $(1, 0)$ and $(0, 1)$ in the xy plane), then it is not unreasonable that other solutions can be formed by the linear combination $c_1 y_1 + c_2 y_2$. This is the plain-language version of the concept of linearly independent functions and basis in a function space.

Definition 17 (Linearly independent functions). Two functions $f(x)$ and $g(x)$ are called linearly independent on an interval I where they are defined if

$$k_1 f(x) + k_2 g(x) = 0 \text{ everywhere on } I \text{ implies } k_1 = k_2 = 0$$

You can find the proof of the above theorems in EK2.7. Details of the proof are not required for this course. However, before picking up the pen to write down solutions to a practical problem, you may want to think first whether the solution actually exists to begin with.

8.3 Solution concepts for nonhomogeneous ODEs

We now move on to the nonhomogeneous linear ODE in the form of

$$\boxed{y^{''} + p(x) y^{'} + q(x) y = r(x)} \tag{26}$$

where $r(x) \neq 0$.

Suppose that y_p satisfies (26) so that

$$y_p^{''} + p(x) y_P^{'}(t) + q(x) y_p = r(t) \tag{27}$$

such a function y_p is called a *particular* solution to (26).

Let y_h be a homogeneous solution which satisfies

$$y_h^{''} + p(x) y_h^{'} + q(x) y_h = 0 \tag{28}$$

Adding (27) to (28) gives

$$(y_p + y_h)^{''} + p(x)(y_p + y_h)^{'} + q(x)(y_p + y_h) = r(t)$$

So the function $y_p + y_h$ also satisfies (26), i.e., **the sum of a particular solution with a homogeneous solution is a new particular solution**.

Conversely, take any two particular solutions to (26), say, y_{p_1} and y_{p_2}. Consider the new function $y_d := y_{p_1} - y_{p_2}$. Easy manipulation gives that y_d satisfies

$$y_d^{''} + p(x) y_d^{'} + q(x) y_d = 0$$

i.e, y_d is a homogeneous solution. Now look at the decomposition:

$$y_{p_1} = y_{p_2} + (y_{p_1} - y_{p_2}) = y_{p_2} + y_d$$

We have shown that **any two particular solutions differ by a homogeneous solution**. This indicates that *all* particular solutions to (26) can be generated by adding homogeneous solutions to one particular solution.

We thus have the following general solution concept for linear ODEs:

1. Find any particular solution y_p

2. Find the homogeneous solution y_h and then let

$$\boxed{y(x) = y_p(x) + y_h(x)} \tag{29}$$

8.4 Solving nonhomogeneous ODEs

In this subsection I give you three ways to solve second-order nonhomogeneous ODEs. The first is easier and solves constant-coefficient ODEs of the form

$$y^{''}+ay^{'}+by=r\left(x\right)$$

where $r\left(x\right)$ is an exponential function, a power of x, a sinusoidal, or sums of products of such functions.

The second works for the general case

$$y^{''}+p\left(x\right)y^{'}+q\left(x\right)y=r\left(x\right) \tag{30}$$

but can be a bit more tedious to compute.

The third one works for general variable-coefficient ODEs, and is usually considered the standard method for solving ODEs.

8.4.1 Method of undetermined coefficients

Functions such as e^{ax}, x^3, $\sin 10x$, and $\cos 10x$ have derivatives similar to themselves. If $r\left(x\right)$ falls in this category of functions, we can try choosing the particular solution y_p that has a form similar to $r\left(x\right)$, but with coefficients that are yet to be determined. For instance, consider

$$y^{''}+y=3x^2$$

Here $r\left(x\right)=3x^2$. Let us try $y_p=K_2x^2+K_1x+K_0$. This is in the same class of functions as $r\left(x\right)$. We have

$$y_p^{''}+y_p=2K_2+K_2x^2+K_1x+K_0$$

The result has to match the right hand side quantity $3x^2$. Hence

$$K_0=-6$$
$$K_1=0$$
$$K_2=3$$

which shall give the coefficients for the particular solution y_p.

This is the method of undetermined coefficients. Table 1 will guide you for the choice of y_p

term in $r\left(x\right)$	choice for y_p
$ke^{\alpha x}$	$Ce^{\alpha x}$
kx^n	$K_nx^n+\ldots K_1x+K_0$
$k\cos\omega x$ and/or $k\sin\omega x$	$M\cos\omega x+N\sin\omega x$
$ke^{\alpha x}\cos\omega x$ and/or $ke^{\alpha x}\sin\omega x$	$e^{\alpha x}\left(M\cos\omega x+N\sin\omega x\right)$

Table 1: Method of undetermined coefficients

Remark 18. There are some special cases that can be solved by additional modification rules during choice of the solution structure. It works as follows: if a term in your choice for y_p happens to be a solution of the homogeneous ODE, then multiply this term by x^k where k is the multiplicity of the root.

Example 19 (Method of undetermined coefficients–special case)**.** Consider the problem

$$y^{''} + 3y^{'} + 2.25y = -10e^{-1.5x}$$

The characteristic equation is

$$\lambda^2 + 3\lambda + 2.25 = 0 \Longrightarrow \lambda_1 = \lambda_2 = -1.5$$

So we have a double root, i.e., the multiplicity of the root is two.

The homogeneous solution is easy to get. It should be

$$y_h = c_1 e^{-1.5x} + c_2 x e^{-1.5x}$$

For the nonhomogeneous ODE, if we choose

$$y_p = Ce^{-1.5x}$$

(this is the standard choice according to Table 1), then

$$y_p^{''} + 3y_p^{'} + 2.25y_p = \left(2.25C - 4.5C + 2.25C\right) e^{-1.5x} = 0$$

regardless of the value of C. So $y_p = Ce^{-1.5x}$ does not work. Instead, following Remark 18, we should select

$$\begin{aligned} y_p &= Cx^2 e^{-1.5x} \\ \Longrightarrow y_p^{'} &= Ce^{-1.5x}\left(2x - 1.5x^2\right) \\ y_p^{''} &= Ce^{-1.5x}\left[2 - 6x + 2.25x^2\right] \end{aligned}$$

Hence

$$y_p^{''} + 3y_p^{'} + 2.25y_p = 2Ce^{-1.5x}$$

Thus

$$C = -5$$

and the full solution is

$$y = y_h + y_p = c_1 e^{-1.5x} + c_2 x e^{-1.5x} - 5x^2 e^{-1.5x}$$

8.4.2 Method of variation of parameters

We mentioned that the method of undetermined coefficients does not work for general variable-coefficient ODEs (30). Assume that the ODE coefficients are all continuous. Here is the method of variation of parameters that works in general. It is credited to Lagrange,[2] and gives you the following "cookbook" procedure:

- define

  $$W(y_1 y_2) = y_1 y_2^{'} - y_2 y_1^{'}$$

  This is called the **Wronskian** of y_1 and y_2.[3]

- a particular solution of (30) is then

  $$y_p = -y_1 \int \frac{y_2 r}{W} \mathrm{d}x + y_2 \int \frac{y_1 r}{W} \mathrm{d}x$$

  where y_1 and y_2 form a basis of solutions of the homogeneous ODE

  $$y^{''} + p(x) y^{'} + q(x) y = 0$$

Example 20. Use the method of variation of parameters to solve Example 19.

Idea of the method For homogeneous solutions, the solution has the form $y_h = c_1 y_1(x) + c_2 y_2(x)$. We know that the particular and general solutions are a bit more complex. Let us replace the constants c_1 and c_2 (the coefficients of the solution) with some more complex functions, and set

$$y_p(x) = u(x) y_1(x) + v(x) y_2(x) \tag{31}$$

This is changing the parameters of the homogeneous solution and keeping the basis functions y_1 and y_2 (hence the name "variation of parameters"). We can substitute the result to the nonhomogeneous ODE (30) to determine $u(x)$ and $v(x)$. Direct substitution is tedious as

$$y_p^{'} = u^{'} y_1 + u y_1^{'} + v^{'} y_2 + v y_2^{'}$$

It turns out we can make the simplification by constraining

$$u^{'} y_1 + v^{'} y_2 = 0$$

which, combined with (31) and (30), shall give the final solution.

[2] Joseph Louis Lagrange (1736-1813) made significant contributions in calculus of variations, mechanics, differential equations, approximation theory, and number theory. He got his first professorship when he was 19 (at the Military Academy of Turin).

[3] Introduced by Józef Hoene-Wronski (1812), a Polish Messianist philosopher who worked in many fields of knowledge, not only as philosopher but also as mathematician, physicist, inventor, lawyer, and economist (from Wikipedia).

8.4.3 Power series method

The power series method is the **standard method for solving linear ODEs with variable coefficients**.

Recall a power series is of the form

$$\sum_{m=0}^{\infty} a_m (x - x_o)^m = a_0 + a_1 (x - x_o) + a_2 (x - x_o)^2 + \dots \tag{32}$$

and in particular, if $x_o = 0$, $\sum_{m=0}^{\infty} a_m x^m$. The reason that the power series method works is that other functions can be expressed as power series expansions. For instance

$$\begin{aligned}
\frac{1}{1-x} &= \sum_{m=0}^{\infty} x^m = 1 + x + x^2 + \dots, \quad (|x| < 1) \\
e^x &= \sum_{m=0}^{\infty} \frac{x^m}{m!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \\
\cos x &= \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \\
\sin x &= \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots
\end{aligned}$$

The solution technique is best explained by an example.

Example 21. Solve

$$y^{'} - y = 0$$

Solution: Let

$$y = \sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + \dots \tag{33}$$

then

$$y^{'} = a_1 + 2a_2 x + \dots = \sum_{m=0}^{\infty} m a_m x^{m-1}$$

Putting the above into $y^{'} - y = 0$ gives

$$a_1 - a_0 = 0,\ 2a_2 - a_1 = 0,\ 3a_3 - a_2 = 0,\ \dots$$

Hence

$$a_1 = a_0,\ a_2 = \frac{a_0}{2!},\ a_3 = \frac{a_0}{3!},\ \dots$$

and

$$y = a_0 + a_0 x + \frac{a_0}{2!} x^2 + \frac{a_0}{3!} x^3 + \dots = a_0 e^x \tag{34}$$

In the last example we were lucky to be able to reduce the infinite sum to a finite length function. When writing infinite sums in general, we have to be careful about the convergence of the sum.

For (32), the n**th partial sum** is

$$S_n(x) = a_0 + a_1(x - x_o) + \cdots + a_n(x - x_o)^n$$

The remaining expression

$$R_n(x) = a_{n+1}(x - x_o)^{n+1} + a_{n+2}(x - x_o)^{n+2} + \dots$$

is called a **remainder** after $a_n(x - x_o)^n$.

For different n's, $S_0(x),\ S_1(x),\ \dots, S_n(x)$ form a sequence. The series (32) is called convergent at $x = x_1$ if

$$\lim_{n\to\infty} S_n(x_1) = S(x_1)$$

and we write

$$S(x_1) = \sum_{n=0}^{\infty} a_n(x_1 - x_o)^n$$

If the sequence diverges at $x = x_1$, the series is called divergent. For instance

$$\sum_{m=0}^{\infty} m!x^m = 1 + x + 2x^2 + \dots$$

is divergent if $x \neq 0$.

Notice that (33) is always convergent at $x = 0$. So it makes sense to talk about the region of convergence. The radius of convergence is defined by

$$\max R:\ \forall x: |x - x_0| < R,\ \text{the series is convergent at } x \tag{35}$$

Theorem 22 (Computation of the radius of convergence). *The radius of convergence in (35) is given by*

$$R = \frac{1}{\lim_{m\to\infty} \sqrt[m]{|a_m|}}$$

or equivalently,

$$R = \frac{1}{\lim_{m\to\infty} \left|\frac{a_{m+1}}{a_m}\right|} \tag{36}$$

To develop some intuitions, you may recall the **ratio test for convergence** from your undergraduate calculus class,[4] that

$$S = \sum_{k=0}^{\infty} u_k$$

[4]See, e.g., "W. Rudin, Principles of Mathematical Analysis".

converges if

$$\rho = \lim_{k\to\infty}\left|\frac{u_{k+1}}{u_k}\right| < 1$$

and diverges if $\rho > 1$. Using the ratio test, the series

$$S(x) = \sum_{m=0}^{\infty} a_m (x - x_o)^m$$

converges if

$$\lim_{m\to\infty}\left|\frac{a_{m+1}(x-x_o)^{m+1}}{a_m (x-x_o)^m}\right| = |x - x_o| \lim_{m\to\infty}\left|\frac{a_{m+1}}{a_m}\right| < 1$$

in other words [using the introduced notation (36)]

$$|x - x_o| < R \Leftrightarrow x \in (x_o - R, x_o + R)$$

For the case in Example 21, we have

$$R = \frac{1}{\lim_{m\to\infty}\left|\frac{\frac{a_0}{(m+1)!}}{\frac{a_0}{m!}}\right|} = \infty$$

Hence the sequence is convergent in the entire domain. This matches with the final result in (34).

Fact 23 (Existance of power series solutions). *Consider*

$$y'' + p(x)\,y' + q(x)\,y = r(x)$$

If q, p, and r have power series representations with positive radius of convergence (called analytic) in the domain I of x, then power series solutions exist.

Some more definitions related to existence and convergence of power series solutions:

Definition 24. The ODE

$$y'' + p(x)\,y' + q(x)\,y = r(x)$$

is called **regular** about a point $x = x_o$ if $p(x)$, $q(x)$, and $r(x)$ can be expanded in a power series about $x = x_o$. x_o in this case is called a regular point. Otherwise, x_o is a **singular** point.

If x_o is a singular point but $(x - x_o)\,p(x)$ and $(x - x_o)^2\,q(x)$ have power series expansions at $x = x_o$, then x_o is called a **regular singular** point.

If x_o is a singular point and $(x - x_o)\,p(x)$ and $(x - x_o)^2\,q(x)$ do not have power series expansions at $x = x_o$, then x_o is called a **irregular singular** point.

Example 25. $x_o = 0$ is a regular point for $y'' - y = 0$.

For $x^2y'' + (x^2 + x)y' - y = 0$, we have $p(x) = 1 + \frac{1}{x}$, $q(x) = -1/x^2$. $x_o = 0$ is a singular point. But $xp(x)$ and $x^2q(x)$ have valid power series expansions at 0. So $x_o = 0$ is a regular singular point.

$x_o = 0$ is a irregular singular point for $x^3y'' + y = 0$. (Exercise)

8.4.3.1 Legendre's equation

Example 26. Use power series method to solve the Legendre's differential equation

$$\boxed{\left(1-x^2\right)y^{''}-2xy^{'}+n\left(n+1\right)y=0} \tag{37}$$

Obtain also the radius of convergence.

Solution: First of all, the normalized standard form of the equation is

$$y^{''}-\frac{2x}{1-x^2}y^{'}+\frac{n\left(n+1\right)}{1-x^2}y=0$$

$x_o=0$ is a regular point for both $-\frac{2x}{1-x^2}$ and $\frac{n(n+1)}{1-x^2}$. Hence we can perform Taylor expansion and let

$$y=\sum_{m=0}^{\infty}a_m x^m=a_0+a_1x+a_2x^2+\ldots$$

Then

$$y^{'}=\sum_{m=0}^{\infty}a_m m x^{m-1}$$

and

$$y^{''}=\sum_{m=0}^{\infty}a_m m\left(m+1\right)x^{m-2}$$

Substituting the above three equations into the Legendre's ODE (37) yields, after some algebra,

$$\sum_{m=0}^{\infty}a_m m\left(m-1\right)x^{m-2}+\sum_{m=0}^{\infty}\left[n\left(n+1\right)a_m-2a_m m-a_m m\left(m-1\right)\right]x^m=0 \tag{38}$$

Notice that the first two coefficients in $\sum_{m=0}^{\infty}a_m m\left(m-1\right)x^{m-2}$ are always zero. By a change of variable, we have

$$\begin{aligned}\sum_{m=0}^{\infty}a_m m\left(m-1\right)x^{m-2}&=\sum_{m=2}^{\infty}a_m m\left(m-1\right)x^{m-2}\\&=\sum_{k=0}^{\infty}a_{k+2}\left(k+2\right)\left(k+1\right)x^k\\&=\sum_{m=0}^{\infty}a_{m+2}\left(m+2\right)\left(m+1\right)x^m\end{aligned}$$

(38) then becomes

$$\sum_{m=0}^{\infty}\left\{a_{m+2}\left(m+2\right)\left(m+1\right)+\left[n\left(n+1\right)a_m-2a_m m-a_m m\left(m-1\right)\right]\right\}x^m=0$$

We thus need

$$a_{m+2} = -\frac{n(n+1) - 2m - m(m-1)}{(m+2)(m+1)} a_m$$
$$= -\frac{(n-m)(n+m+1)}{(m+2)(m+1)} a_m, \quad m = 0, 1, 2, \dots \tag{39}$$

The equation gives a recursive formula to compute a_{m+2} from a_m, except for a_0 and a_1, which are left as arbitrary constants. We can thus write

$$y(x) = a_0 y_1(x) + a_1 y_2(x)$$

where coefficients of $a_0 y_1(x)$ are from (39) initialized by a_0; and those of $a_1 y_2(x)$ are solely dependent on a_1. To compute the radius of convergence for the sequence $\dots, a_m, a_{m+2}, \dots$, we can use the ratio test and get

$$R = \frac{1}{\lim_{m\to\infty} \left| \frac{a_{m+2}}{a_m} \right|} = \frac{1}{\lim_{m\to\infty} \left| \frac{(n-m)(n+m+1)}{(m+2)(m+1)} \right|} = 1$$

Hence the two linearly independent solutions $y_1(x)$ and $y_2(x)$ both converge for $|x| < 1$.

□

Remark 27. The ODE solution written in the summation form

$$y(x) = \sum_{m=0}^{\infty} a_m (x - x_o)^m$$

is called the series solution of the ODE.

8.4.3.2 Extension: Frobenius method Legendre's equation (37) can be solved by standard power series method. Consider ODEs in the form of

$$\boxed{y'' + \frac{b(x)}{x} y' + \frac{c(x)}{x^2} y = 0} \tag{40}$$

Notice that $x_o = 0$ is a singular point. So the standard Taylor expansion at 0 is no longer valid. But $x_o = 0$ is at least a regular singular point. An extension of the power series method, called the **Frobenius method**, solves ODEs in the form of (40).

The intuition is that, if y has the form $x^r \sum_{m=0}^{\infty} a_m x^m$, then the troublesome $1/x$ and $1/x^2$ *may* get canceled in the ODE (this is only an intuition, not a constraint that the actual y has to satisfy.[5]).

Theorem 28. *Let $b(x)$ and $c(x)$ be any functions that have valid Taylor expansions at $x = 0$ (i.e., analytic at $x = 0$). Then*

$$y'' + \frac{b(x)}{x} y' + \frac{c(x)}{x^2} y = 0$$

[5] In other words, y can be singular at 0.

has at least one solution that can be represented in the form of

$$y(x) = x^r \sum_{m=0}^{\infty} a_m x^m \tag{41}$$

where r can be real or complex and $x_0 \neq 0$.

A second linearly independent solution can be found, e.g., by reduction of order.

To see the intuitions of the method, expand the ODE coefficients to their power series

$$b(x) = \sum_{m=0}^{\infty} b_m x^m = b_0 + b_1 x + b_2 x^2 + \dots$$
$$c(x) = \sum_{m=0}^{\infty} c_m x^m = c_0 + c_1 x + c_2 x^2 + \dots$$

and transform the ODE into the more convenient form

$$x^2 y'' + x b(x) y' + c(x) y = 0$$

Substituting in (41) yields

$$\sum_{m=0}^{\infty} a_m (m+r)(m+r-1) x^{m+r} + \sum_{m=0}^{\infty} b_m x^m \sum_{m=0}^{\infty} a_m (m+r) x^{m+r} + \sum_{m=0}^{\infty} c_m x^m \sum_{m=0}^{\infty} a_m x^{m+r} = 0 \tag{42}$$

Looking at the coefficient for x^r (i.e. $m = 0$), we must have

$$a_0 r (r-1) + b_0 a_0 r + c_0 a_0 = 0$$

As $a_0 \neq 0$, it must be that

$$\boxed{r(r-1) + b_0 r + c_0 = 0} \tag{43}$$

Solving (43) gives values of r and we can proceed to solve other coefficients a_m's in (42).

Example 29 (Euler-Cauchy Equation). Solve

$$x^2 y'' + x y' - y = 0$$

Solution: Letting $y(x) = x^r \sum_{m=0}^{\infty} a_m x^m$ gives

$$\sum_{m=0}^{\infty} a_m (m+r)(m+r-1) x^{m+r} + \sum_{m=0}^{\infty} a_m (m+r) x^{m+r} - \sum_{m=0}^{\infty} a_m x^{m+r} = 0 \tag{44}$$

When $m = 0$, we need the coefficients of x^r to be zero, namely

$$r(r-1) + r - 1 = 0$$

whose solution is

$$r_1 = 1,\ r_2 = -1$$

For $r_1 = 1$, (44) becomes

$$\sum_{m=0}^{\infty}\left[a_m(m+1)mx^{m+1} + a_m(m+1)x^{m+1} - a_m x^{m+1}\right] = \sum_{m=0}^{\infty} a_m\left(m^2+2m\right)x^{m+1}$$
$$= a_1(1+2)x^2 + a_2(4+4)x^3 + \cdots = 0$$

As $m \geq 0$, it must be that $a_1 = a_2 = \cdots = 0$. Hence one solution is

$$y_1 = x^{r_1} = x$$

For $r_2 = -1$, analogous analysis gives that $a_0 \neq 0$ and $a_2 \neq 0$. Hence

$$y_2 = x^{-1}\left(a_0 + a_2x^2\right) = a_0x^{-1} + a_2x$$

The general solution is thus

$$y = A_1y_1 + A_2y_2 = Ax^{-1} + Bx$$

□

Remark 30. Usually negative and zero values of r are not convenient for finding the second linearly independent solution (see the example above). Performing reduction of order is usually faster in practice, as we will do in the next example.

Example 31. Solve

$$\left(x^2 - x\right)y'' - xy' + y = 0 \tag{45}$$

Solution: Let $y(x) = x^r\sum_{m=0}^{\infty} a_m x^m$. (42) becomes

$$\left(x^2-x\right)\sum_{m=0}^{\infty} a_m(m+r)(m+r-1)x^{m+r-2} - \sum_{m=0}^{\infty} a_m(m+r)x^{m+r} + \sum_{m=0}^{\infty} a_m x^{m+r} = 0$$

which is equivalent to

$$-\sum_{m=0}^{\infty} a_m(m+r)(m+r-1)x^{m+r-1} + \sum_{m=0}^{\infty}\left[a_m(m+r)(m+r-1) - a_m(m+r) + a_m\right]x^{m+r} = 0 \tag{46}$$

The lowest power is x^{r-1}. Letting its coefficient to be 0 gives

$$-a_0r(r-1) = 0 \Leftrightarrow r(r-1) = 0$$

So $r = 1$ or 0.

Take $r = r_1 = 1$. (46) becomes

$$-\sum_{m=0}^{\infty} a_m(m+1)mx^m + \sum_{m=0}^{\infty} a_m m^2 x^{m+1} = 0$$

Note that

$$-\sum_{m=0}^{\infty} a_m (m+1) m x^m = -\sum_{m=1}^{\infty} a_m (m+1) m x^m = -\sum_{s=0}^{\infty} a_{s+1} (s+2)(s+1) x^{s+1}$$

and

$$\sum_{m=0}^{\infty} a_m m^2 x^{m+1} = \sum_{s=0}^{\infty} a_s s^2 x^{s+1}$$

Hence

$$\sum_{s=0}^{\infty} \left[a_s s^2 - a_{s+1} (s+2)(s+1) \right] x^{s+1} = 0$$

We thus need

$$a_{s+1} = \frac{s^2}{(s+1)(s+2)} a_s, \;\; s = 0, 1, \ldots$$

Hence for $a_0 \neq 0$, we have $a_1 = 0$, $a_2 = 0$,... Taking $a_0 = 1$, we have

$$y_1 = x^{r_1} a_0 = x$$

For the second linearly independent solution, we perform reduction of order and let

$$y_2 = y_1 u = xu$$

Then

$$y_2^{'} = u + xu^{'}$$
$$y_2^{''} = 2u^{'} + xu^{''}$$

(45) then becomes, after simplification,

$$\frac{u^{''}}{u^{'}} = -\frac{x-2}{x^2 - x} = -\frac{2}{x} + \frac{1}{1-x}$$

Integrating gives

$$\ln u^{'} = \ln \left| \frac{x-1}{x^2} \right|$$

One solution is

$$u^{'} = \frac{x-1}{x^2} = \frac{1}{x} - \frac{1}{x^2}$$

and hence

$$u = \ln x + \frac{1}{x}$$
$$y_2 = x \ln x + 1$$

y_1 and y_2 are now linearly independent, and form a basis for the general solution of the ODE.

Exercise 32. Find the first four terms in the series solutions for

$$\left(x^2 + x\right) \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - \left(x^2 - 2\right) \frac{\mathrm{d}y}{\mathrm{d}x} - (x+2) y = 0$$

8.4.3.3 Bessel's equation

One important ODE in applied mathematics is Bessel's equation

$$\boxed{x^2y'' + xy' + \left(x^2 - v^2\right)y = 0}$$

It is a special case of (40), and hence has *a* solution of the form

$$y(x) = x^r \sum_{m=0}^{\infty} a_m x^m,\ a_0 \neq 0$$

The first particular solution Following the procedures (41-43) in the last section, we can get

$$r(r-1) + r - v^2 = 0$$

in other words,

$$r_1 = v\,(\geq 0)\,,\ r_2 = -v$$

You should be able to show that the solution corresponding to r_1 satisfies

$$a_{2m} = -\frac{1}{2^2 m\,(v+m)} a_{2m-2}$$

and hence

$$\boxed{a_{2m} = \frac{(-1)^m a_0}{2^{2m} m!\,(v+1)(v+2)\dots(v+m)}}$$

We need to choose a_0. There are two cases:

- If v is an integer, say $v = n$, choosing

  $$a_0 = \frac{1}{2^n n!}$$

  is convenient as it gives a simplified solution

  $$a_{2m} = -\frac{(-1)^m}{2^{2m+n} m!\,(n+m)!}$$

  A particular solution of Bessel's equation, denoted as $J_n(x)$, is thus

  $$J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n} m!\,(n+m)!} \tag{47}$$

  This is called the Bessel function of the first kind of *(integer) order* n. The series converges for all x, by the ratio test.

- If $v \geq 0$ is not limited to be an integer, choosing

$$a_0 = \frac{1}{2^v \Gamma(v+1)}$$

where $\Gamma(\cdot)$ is the gamma function defined by

$$\Gamma(v+1) = \int_0^\infty e^{-t} t^v \mathrm{d}t$$

gives the simplified solution

$$a_{2m} = \frac{(-1)^m}{2^{2m+v} m! \Gamma(v+m+1)}$$

Here the simplification comes from the nice properties of gamma functions, that

$$\Gamma(v+1) = v\Gamma(v)$$
$$\Gamma(n+1) = n!$$

The final particular solution, called the Bessel function of the first kind of *(general) order* v, is

$$J_v(x) = x^v \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+v} m! \Gamma(v+m+1)} \tag{48}$$

The second particular solution We need a second linearly independent solution to form the general solution. Again, there are two cases:

- If v is not an integer,

$$J_{-v}(x) = x^{-v} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+v} m! \Gamma(v+m+1)}$$

can directly be used as it is linearly independent with $J_v(x)$ in (48). Unfortunately, we cannot do this to (47), as it turns out that $J_{-n}(x)$ is linearly dependent with $J_n(x)$ due to

$$J_{-n}(x) = (-1)^n J_n(x)$$

- If $v\,(\geq 0)$ is not limited to be an integer, choose instead

$$Y_v(x) = \frac{1}{\sin v\pi} \left[J_v \cos v\pi - J_{-v}(x)\right] \tag{49}$$

for linearly independence. (49) is called the Bessel function of the second kind of order v.

From (49) and (48), the general solution of Bessel's equation for all values of v and $x > 0$ is

$$y(x) = C_1 J_v(x) + C_2 Y_v(x)$$

9 Review

For first-order ODEs we have learned that

- The solution of $y^{'} + p(x) y = 0$ is $y = ce^{-\int p(x)\mathrm{d}x}$, by integration of a separable ODE.
- For a general 1st-order ODE $y^{'} = f(x, y)$, a particular case–$y^{'} = f(y/x)$–can be solved by *reduction to a separable ODE*, via introducing $u = y/x$.
- A nonhomogeneous first-order ODE $y^{'} + p(x) y = r(x)$ has the solution

  $$y(x) = e^{-h}\left(\int e^{h} r \mathrm{d}x + c\right), \ h = \int p(x) \mathrm{d}x$$

  This is done by finding an integrating factor $F(x)$ such that

  $$F(x) y^{'} + F(x) p(x) y = F(x) r(x)$$
  $$F^{'}(x) = F(x) p(x)$$

For second-order ODEs we have learned:

- Homogeneous equation
  - $y^{''} + ay^{'} + by = 0$ always have solutions. There are only three possible cases, depending on the value of $a^2 - 4b$.
    - case I: the characteristic equation has two different real roots
    - case II: the characteristic equation has two complex roots
    - case III: the characteristic equation has two repeated roots. We have $y_1 = e^{\lambda x}$ and found the other linearly independent solution by letting $y_2 = u(x)y_1$ and *reduction of order.*
  - the more general case $y^{''} + p(x) y^{'} + q(x) y = 0$ always has solutions if $p(x)$ and $q(x)$ are continuous on the specified interval of x. In addition, the solution structure is

    $$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

- Nonhomogeneous equation $y^{''} + p(x) y^{'} + q(x) y = r(x)$
  - solution structure:

    $$y(x) = y_p(x) + y_h(x)$$

  - one particular solution can be obtained via
    - method of undetermined coefficients: suitable if $p(x)$ and $q(x)$ are constants
    - method of variation of parameters: a cookbook procedure if $y_1(x)$ and $y_2(x)$ have been found; works if $p(x)$ and $q(x)$ are continuous functions
    - power series method: solution exists if $p(x)$ $q(x)$ and $r(x)$ have power series representations with positive radius of convergence in the domain of x
- The Frobenius method provide solutions to several important special functions such as Legendre's equation and Bessel equation.

10 Higher-order linear ODEs

The basic ideas for solving second-order linear ODEs all apply to the case of higher-order linear ODEs.

A general nth-order ODE has the form of

$$F\left(x, y, y', \ldots, y^{(n)}\right) = 0$$

The ODE is called linear if it can be written

$$y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + \cdots + p_1(x)\, y' + p_0(x)\, y = r(x) \tag{50}$$

This is the standard form of an nth-order linear ODE.

10.1 Homogeneous linear ODEs–general forms and solution concepts

10.1.1 Main results

If $r(x) = 0$ in (50) then the simplified case

$$y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + \cdots + p_1(x)\, y' + p_0(x)\, y = 0 \tag{51}$$

is called homogeneous.

Theorem 33 (Fundamental theorem for homogeneous linear ODEs). *For a homogeneous linear ODE (51), sums and constant multiples of solutions on some open interval I are again solutions on I.*

This is a natural extension of the fundamental theorem for 2nd-order homogeneous linear ODEs. The proof is analogous to that for Theorem 12.

The following theorems are also parallel to the results for 2nd-order linear ODEs.

An initial value problem for the ODE (51) consists of (51) and n initial conditions

$$y(x_0) = K_0,\ y'(x_0) = K_1, \ldots, y^{(n-1)}(x_0) = K_{n-1}$$

Theorem 34 (Existence and uniqueness theorem for initial value problems). *If the coefficients of (51), namely, $p_0(x), \ldots, p_{n-1}(x)$ are continuous on the open interval I and x_0 is in I, then the initial value problem has a unique solution $y(x)$ on I.*

□

Theorem 35 (General solution, basis, particular solution). *A general solution of the homogeneous linear ODE (51) on an open interval I has the form of*

$$y(x) = c_1 y_1(x) + \cdots + c_n y_n(x)$$

where y_1, ..., y_n form a basis of solutions of (51), namely, these solutions are linearly independent on I.

A particular solution of (51) is obtained if we assign specific values to the n constants $c_1, \ldots, c_n$.

□

If initial conditions are given, the constants $c_1, \ldots, c_n$ are unique.

Theorem 36 (Linear independence of functions). *Let $y_1(x), \ldots, y_n(x)$ be n functions defined on some interval I. The n functions are called linearly independent on I if the equation*

$$k_1 y_1(x) + \cdots + k_n y_n(x) = 0,\ x \in I$$

implies that

$$k_1 = k_2 = \cdots = k_n = 0$$

10.1.2 Checking linear independence–Wronskian

For simple cases, linear independence can be checked by inspection. A more systematic way of doing this uses the Wronskian of $y_1, \ldots, y_n$ defined as the determinant

$$W(y_1, \ldots, y_n) = \det \begin{bmatrix} y_1 & y_2 & \ldots & y_n \\ y_1' & y_2' & \ldots & y_n' \\ \vdots & \vdots & \ldots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \ldots & y_n^{(n-1)} \end{bmatrix}$$

It should be kept in mind that W depends on x as y_1, y_2, ..., y_n do.

Theorem 37 (Conditions of linear independent functions). *n functions $y_1, \ldots, y_n$, defined on an open interval I, are linearly independent on I if their Wronskian is nonzero for some $x = x_0$ in I. They are linearly dependent if and only if their Wronskian is zero for all $x \in I$.*

Proof. (main ideas) Let $y_1, \ldots, y_n$ be linear dependent solutions on I. Then there exist constants $k_1, \ldots, k_n$ not all zero, such that for all $x \in I$

$$k_1 y_1 + \cdots + k_n y_n = 0$$

Differentiating gives

$$\begin{aligned} k_1 y_1' + \cdots + k_n y_n' &= 0 \\ &\vdots \\ k_1 y_1^{(n-1)} + \cdots + k_n y_n^{(n-1)} &= 0 \end{aligned}$$

So the equation set

$$\begin{bmatrix} y_1 & y_2 & \ldots & y_n \\ y_1' & y_2' & \ldots & y_n' \\ \vdots & \vdots & \ldots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \ldots & y_n^{(n-1)} \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix} = 0 \tag{52}$$

has nonzero solutions. Hence it must be that

$$\det \begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_1^{'} & y_2^{'} & \cdots & y_n^{'} \\ \vdots & \vdots & \cdots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{bmatrix} = 0$$

for any $x \in I$.

To prove the other direction, that $W = 0$ indicates linear dependence, we notice first that if $W = 0$ for some $x = x_0$, we can find a nonzero solution $k_1^*, k_2^*, \ldots, k_n^*$ of (52). Let $y^* = k_1^* y_1 + k_2^* y_2 + \ldots k_n^* y_n$. Then $y^*(x_0), (y^*)^{'}(x_0), \ldots, (y^*)^{(n-1)}(x_0)$ are all zero from (52). So y^* is a solution to the homogeneous equation, with the initial conditions $y^*(x_0) = 0,\ (y^*(x_0))^{'} = 0, \ldots$. But on the other hand a trivial solution is $y \equiv 0$ for any x. Because the initial condition problem has one unique solution, it must be that

$$y^* = 0$$

namely, $y_1, \ldots, y_n$ are linearly dependent.

The case for linear independence is analogous. □

□

Example 38. $y_1 = x^2$, $y_2 = x$, $y_3 = 2x$ are linearly dependent on any interval. $y_1 = x$, $y_2 = x^2$, $y_3 = x^3$ are linearly independent on any interval as

$$W = \det \begin{bmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{bmatrix} = 12x^3 + 2x^3 - 12x^3 = 2x^3$$

which is zero only when $x = 0$.

Theorem 39 (Existence of a general solultion). *If the coefficients $p_0(x)$, $p_1(x)$, ..., $p_{n-1}(x)$ of (51) are continuous on an open interval I, then (51) has a general solution on I.*

□

Theorem 40 (General solution includes all solutions). *If (51) has continuous coefficients on an open interval I then every solution $y = Y(x)$ on I is of the form*

$$Y(x) = C_1 y_1(x) + \cdots + C_n y_n(x)$$

where y_1, ..., y_n is a basis of solutions of (51) on I and $C_1, \ldots, C_n$ are suitable constants.

10.2 Homogeneous linear ODEs with constant coefficients

For

$$y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + \cdots + p_1(x)\, y' + p_0(x)\, y = 0 \tag{53}$$

now we know the structure of the homogeneous solution is

$$y(x) = c_1 y_1(x) + \cdots + c_n y_n(x)$$

we need to find the basis functions $y_1, \ldots, y_n$.

Similar to the case of 2nd-order ODEs, we substitute $y = e^{\lambda x}$ to (53) to get the characteristic equation

$$\lambda^{(n)} + a_{n-1}\lambda^{(n-1)} + \cdots + a_1\lambda + a_0\lambda = 0 \tag{54}$$

If λ is a root of (54), then $y = e^{\lambda x}$ is a solution of (53).

Using numeric software such as MATLAB, you can solve the characteristic equation (the command is **roots**([1, a_{n-1}, a_{n-2}, ..., a_0])).

There are again different cases for λ.

10.2.1 Distinct real roots

If all n roots $\lambda_1, \ldots, \lambda_n$ are real and different, then

$$y_1 = e^{\lambda_1 x},\ y_2 = e^{\lambda_2 x},\ \ldots,\ y_n = e^{\lambda_n x}$$

constitute a basis for the homogeneous solution, and

$$y = c_1 e^{\lambda x} + e^{\lambda_2 x} + \ldots, + e^{\lambda_n x}$$

10.2.2 Simple complex roots

Complex roots always appear in pairs in polynomial equations. If $\lambda = \gamma + i\omega$ is a simple root of the characteristic equation, then so is the complex conjugate $\overline{\lambda} = \gamma - i\omega$. Recall that we prefer real over complex solutions. The corresponding linearly independent solutions are

$$y_1 = e^{\gamma x}\cos\omega x,\ y_2 = e^{\gamma x}\sin\omega x$$

10.2.3 Multiple real roots

If λ is a real root of order m, then m corresponding linearly independent solutions are

$$e^{\lambda x},\ xe^{\lambda x},\ x^2 e^{\lambda x}, \ldots,\ x^{m-1}e^{\lambda x}$$

10.2.4 Multiple complex roots

If $\lambda = \gamma + i\omega$ is a complex double root, then $\gamma - i\omega$ is also a double root. The four linearly independent solutions are

$$e^{\gamma x}\cos\omega x,\ e^{\gamma x}\sin\omega x,\ xe^{\gamma x}\cos\omega x,\ xe^{\gamma x}\sin\omega x$$

If $\gamma + i\omega$ is a triple root (very rare in practice) then there are two more solutions

$$x^2 e^{\gamma x}\cos\omega x,\ x^2 e^{\gamma x}\sin\omega x$$

and so on.

Example 41. Suppose we want to solve the differential equation

$$y^{(5)} + 10y^{(4)} + 54y^{(3)} + 132y^{''} + 137y^{'} + 50y = 0$$

subject to certain initial conditions. The roots of the characteristic equation

$$\lambda^5 + 10\lambda^4 + 54\lambda^3 + 132\lambda^2 + 137\lambda + 50 = 0$$

are

$$-1, -1, -2, -3 \pm 4i$$

Thus, the form of the solution is

$$y_h(t) = e^{-t}\left(c_1 + c_2 t\right) + c_3 e^{-2t} + e^{-3t}\left(c_4\cos(4t) + c_5\sin(4t)\right)$$

The constants $c_1, \cdots, c_5$ may be found by forcing the given initial conditions on the solution above.

Let's summarize our discussion so far.

We want to solve the homogeneous differential equation

$$y^{(n)} + p_{n-1}y^{(n-1)} + \cdots + p_1 y^{'} + p_0 y = 0$$

First make a list of all the roots (including multiplicity) of the characteristic equation as

$$q_1, q_1, \cdots, q_1,\quad q_2, \cdots, q_2,\quad \cdots,\quad \cdots,\quad q_k, q_k, \cdots, q_k, q_k$$

There will be a total of n roots in this list, where n is the order of the differential equation. The homogeneous solution will have n terms in it: one per root. The terms in the free response correspinding to a root q which occurs with multiplicity m are of the form

$$e^{qx}\left[c_1 + c_2 x + \cdots + c_m x^{m-1}\right]$$

If the pole q above is complex, say $q = \gamma \pm i\omega$, it is more convenient to use the expression

$$e^{\gamma x}\cos(\omega x)\left[d_1 + d_3 x + \cdots + d_{2m-1}x^{m-1}\right] + e^{\gamma x}\sin(\omega x)\left[d_2 + d_4 x + \cdots + d_{2m}t^{m-1}\right]$$

Finally, there will be n unknown constants in the homogeneous solution. If initial conditions are given, these are found by enforcing the initials conditions on the solution.

10.2.5 Stability

Beyond the process of obtaining the mathematical formulas, it is very important to analyze and develop the intuition of a solution.

In this section, we study the behavior of the class of homogeneous solutions as $x \to \infty$. If all homogeneous solutions $y_h(x)$ decay to 0 as $x \to \infty$, then from (29) it must be that for a give input function $r(x)$, all particular solutions, regardless of the initial conditions, approach each other. This will be useful in many contexts, to quickly understand how a system behaves.

We shall say that the ODE is *asymptotically stable* if for all initial conditions, the homogeneous solution decays $y_h(x)$ to zero. This concept is suggested by physics, where stability means, roughly speaking, that a small change (a small disturbance) of a physical system at some instant changes the behavior of the system only slightly at all future times t. Stability concepts are basic not just in ODEs but also in engineering and other applications.

Suppose r is a complex number, $r \in \mathbb{C}$, and we decompose it into its real and imaginary parts. Let $\sigma := \mathrm{Real}(r)$ and $\omega := \mathrm{Imag}(r)$. Hence, both σ and ω are real numbers, and $r = \sigma + i\omega$. The exponential function e^{rt} can be expressed as

$$e^{rx} = e^{\sigma x}\left[\cos \omega x + i \sin \omega x\right]$$

Note that the real part of r, namely σ, determines the qualitative behavior ofe^{rt} as $x \to \infty$. Specifically,

- if $\sigma < 0$, then $\lim_{x\to\infty} e^{rt} = 0$
- if $\sigma = 0$, then e^{rx} does not decay to 0, and does not "explode," but rather oscillates, with $|e^{rx}| = 1$ for all x
- if $\sigma > 0$, then $\lim_{x\to\infty} |e^{rx}| = \infty$

When all of the roots of the chracteristic polynomial are distinct, all homogeneous solutions are of the form e^{rx} where r is a root of the characteristic polynomial. Therefore, we see that the roots of the characteristic polynomial determine the qualitative nature of the homogeneous solutions. If the characteristic polynomial has a root q_p of multiplicity $l_p \geq 1$, then the homogeneous solution modes associated the root q_p are

$$e^{q_p x},\ x\, e^{q_p x},\ \cdots,\ x^{(l_p - 1)}\, e^{q_p x}\,.$$

Notice that for any real constants $m < \infty$, and $\sigma < 0$,

$$\lim_{x\to\infty} x^m\, e^{\sigma x} = 0\,.$$

Therefore, the real part of the repeated root q_p, namely σ_p, also determines the qualitative behavior of the homogeneous solution modes associated with this repeated root as $x \to \infty$. Specifically,

- if $\sigma_p < 0$, then $\lim_{x\to\infty} x^m\, e^{q_p x} = 0$ for $m = 0, \cdots l_p - 1$.
- if $\sigma_p = 0$, then $|e^{q_p x}| = 1$ for all x, but
- if $\sigma_p \geq 0$, then $\lim_{x\to\infty} |x^m e^{q_p x}| = \infty$ for $m = 1, \cdots l_p - 1$

where $l_p \geq 1$ is the multiplicity of the root q_p. We summarize all of these results as follows:

- If **all** of the roots, $\{r_i\}_{i=1}^n$, of the characteristic polynomial satisfy

$$\mathrm{Real}(r_i) < 0$$

  then **every** homogeneous solution decays to 0 as $x \to \infty$.

- If **any** of the roots, $\{r_i\}_{i=1}^n$, of the characteristic polynomial satisfy

$$\mathrm{Real}(r_i) \geq 0$$

  then there are homogeneous solutions that **do not** decay to 0 as $x \to \infty$.

Example 42. Recall Example 9 on page 11. Let the applied force be denoted as $u(t)$. If there is no damper, Newton's law gives

$$MS'' + KS = u(t)$$

The homogeneous solution is (derive it yourself)

$$S(t) = A\cos\left(\sqrt{\frac{K}{M}}t\right) + B\sin\left(\sqrt{\frac{K}{M}}t\right) \tag{55}$$

where A and B depends on the initial condition. This is a oscillation function and will not converge to zero for all initial conditions. Hence the system is not asymptotically stable. This is intuitive to understand. If everything is ideal and the floor has no frictions, releasing the mass with the spring having some initial extensions will make the mass oscillate without stop.

If the applied force is $u(t) = E\cos\left(\sqrt{\frac{K}{M}}t\right)$, we can apply the method of undetermined coefficients to solve the particular solution. Note that the input $u(t)$ is in the form of a basis function in the homogeneous solution (55). We need to use the modification rule and let

$$S_p(t) = Ct\cos\left(\sqrt{\frac{K}{M}}t\right) + Dt\sin\left(\sqrt{\frac{K}{M}}t\right)$$

You should be able to find the formula for C and D yourself. Regardless of the process, you see that $S_p(t)$ will keep oscillating with the oscillation magnitude larger and larger. As $S = S_p + S_h$, the overall response of the mass will eventually go to infinity.

So, we applied a bounded input to the system but obtained a unbounded output in the end. Not very desired for a mechanical system.

Consider, on the other hand, applying

$$u(t) = E\cos\left(\sqrt{\frac{K}{M}}t\right) - bS'$$

namely, in the input we *feedback* the velocity information to the system. Then we need to solve instead

$$MS^{''} + bS^{'} + KS = E\cos\left(\sqrt{\frac{K}{M}}t\right)$$

If b, K and M are all positive, you can verify that the homogeneous solution will converge to zero. So the new system is stable. And a particular solution will be in the form of

$$S_p = C\cos\left(\sqrt{\frac{K}{M}}t\right) + D\sin\left(\sqrt{\frac{K}{M}}t\right) \tag{56}$$

which is bounded and in the same class of function as the input. In this example $E\cos\left(\sqrt{\frac{K}{M}}t\right)$ is the desired response. If we design the value of b well, we can make (56) very similar to the desired response.

By adding the term of $-bS^{'}$ in the design, we have *controlled* the mass-spring system from an unstable one to a stable one, and enabled the system to behave in ways that we want. This is the power of *feedback control*.

10.3 Homogeneous linear ODE with non-constant coefficients

Non-constant ODEs (even if they are linear) are more difficult to solve. I give you one example to demonstrate the solution techniques.

10.3.1 Euler-Cauchy equation example

Example 43 (Euler-Cauchy equation). Solve

$$x^3y^{(3)} - 3x^2y^{''} + 6xy^{'} - 6y = 0$$

We no longer can obtain a characteristic equation. Instead, substitute $y = x^m$ and its derivatives of the ODE gives

$$x^3m\left(m-1\right)\left(m-2\right)x^{m-3} - 3x^2m\left(m-1\right)x^{m-2} + 6xmx^{m-1} - 6x^m = 0$$

namely

$$\left[m\left(m-1\right)\left(m-2\right) - 3m\left(m-1\right) + 6m - 6\right]x^m = 0$$

Either $x = 0$ or the following has to hold

$$m\left(m-1\right)\left(m-2\right) - 3m\left(m-1\right) + 6m - 6 = 0 \quad (57)$$

You see now, that the key step is to observe that the order of x in individual terms of the ODE always matches with that of the derivative of y. This gives the intuition of using x^m as a solution candidate. The roots of (57) are 1, 2, and 3, giving the basis

$$y_1 = x,\ y_2 = x^2,\ y_3 = x^3$$

Hence the solution is

$$y = c_1x + c_2x^2 + c_3x^3$$

Remark 44. If there are double roots in (57), say $m = 2$ is a double root, then let $y_1 = x^2$ and apply the method of reduction of order to obtain another linearly independent solution $y_2 = uy_1$.

10.4 Nonhomogeneous linear ODEs

The full solution of

$$y^{(n)} + p_{n-1}\left(x\right)y^{(n-1)} + \cdots + p_1\left(x\right)y^{'} + p_0\left(x\right)y = r\left(x\right) \quad (58)$$

is composed of

$$y\left(x\right) = y_h\left(x\right) + y_p\left(x\right)$$

where y_h is the homogeneous solution; y_p is a particular solution of (58). This solution concept is exactly the same as the learned lower-order cases. And the solution techniques are also just extensions of previous special cases. The steps of the method of undetermined coefficients are identical to the

2nd-order case. I give you next an example using the method of variation of parameters. The formula for the particular solution is

$$\boxed{y_p(x) = \sum_{k=1}^{n} y_k(x) \int \frac{W_k(x)}{W(x)} r(x)\,\mathrm{d}x} \tag{59}$$

where $W(x)$ is the Wronskian, and W_k is obtained from W by replacing the kth column of W by the column $[0, 0, \ldots, 0, 1]^T$.

Example 45 (Initial value problem). Solve

$$x^3 y^{(3)} - 3x^2 y'' + 6xy' - 6y = x^4 \ln x$$

This is called the nonhomogeneous Euler-Cauchy equation. Transforming it to the standard form gives

$$y^{(3)} - 3\frac{1}{x}y'' + 6\frac{1}{x^2}y' - 6\frac{1}{x^3}y = \underbrace{x \ln x}_{r(x)}$$

From Example 43, the homogeneous solution is

$$y_h = c_1 x + x_2 x^2 + c_3 x^3$$

To use (59), we need to compute the Wronskian

$$W = \det \begin{bmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{bmatrix} = \det \begin{bmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{bmatrix} = 2x^3$$

and the W_k terms

$$W_1 = \det \begin{bmatrix} 0 & x^2 & x^3 \\ 0 & 2x & 3x^2 \\ 1 & 2 & 6x \end{bmatrix} = x^4,\; W_2 = \det \begin{bmatrix} x & 0 & x^3 \\ 1 & 0 & 3x^2 \\ 0 & 1 & 6x \end{bmatrix} = -2x^3,\; W_3 = \det \begin{bmatrix} x & x^2 & 0 \\ 1 & 2x & 0 \\ 0 & 2 & 1 \end{bmatrix} = x^2$$

Hence the particular solution is

$$\begin{aligned} y_p(x) &= \sum_{k=1}^{n} y_k(x) \int \frac{W_k(x)}{W(x)} r(x)\,\mathrm{d}x \\ &= x \int \frac{x^4}{2x^3} x \ln x \mathrm{d}x - x^2 \int x \ln x \mathrm{d}x + x^3 \int \frac{1}{2x} x \ln x \mathrm{d}x \\ &= \frac{x}{2}\left(\frac{x^3}{3}\ln x - \frac{x^3}{9}\right) - x^2\left(\frac{x^2}{2}\ln x - \frac{x^2}{4}\right) + \frac{x^3}{2}(x \ln x - x) \\ &= \frac{1}{6}x^4\left(\ln x - \frac{11}{6}\right) \end{aligned}$$

Summing up, the full solution to the nonhomogeneous Euler-Cauchy equation is

$$y = y_h + y_p = c_1 x + c_2 x^2 + c_3 x^3 + \frac{1}{6}x^4\left(\ln x - \frac{11}{6}\right)$$

11 References

[EK]: Erwin Kreyszig: Advanced Engineering Mathematics, 10th edition.