Eleven Tools in Feedback Control

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- Basics: Arithmetic of LTI systems, Goals of feedback, Loop shaping, Tradeoffs
- Fundamental limitations
  - Bandwidth
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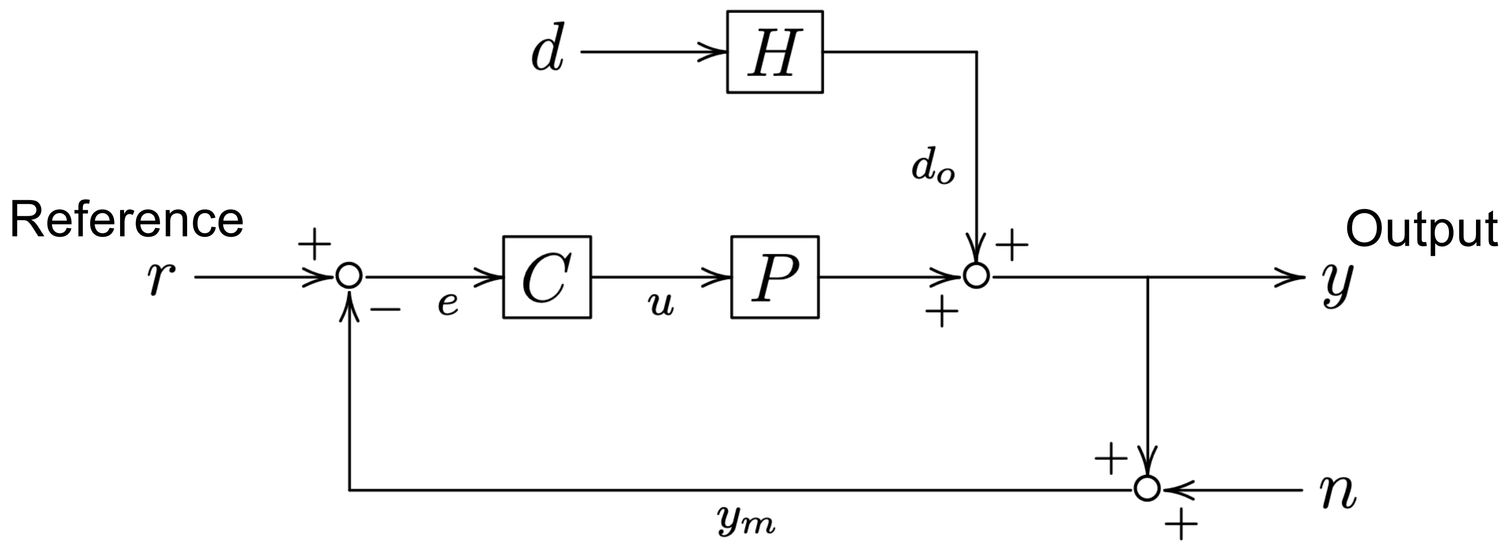
  - Unstable zeros
  - Magnitude-phase relationship
- Practical control engineering
  - Sampling time
  - Delays
  - Time-frequency relationship

#1

Arithmetic of feedback loops



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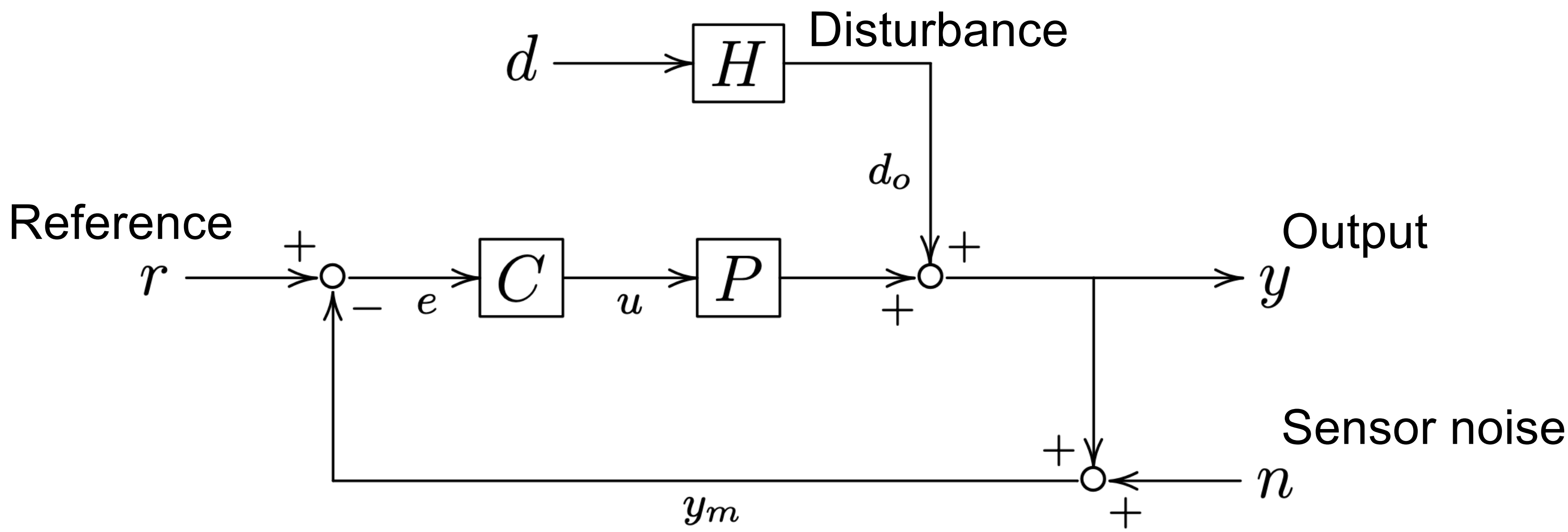

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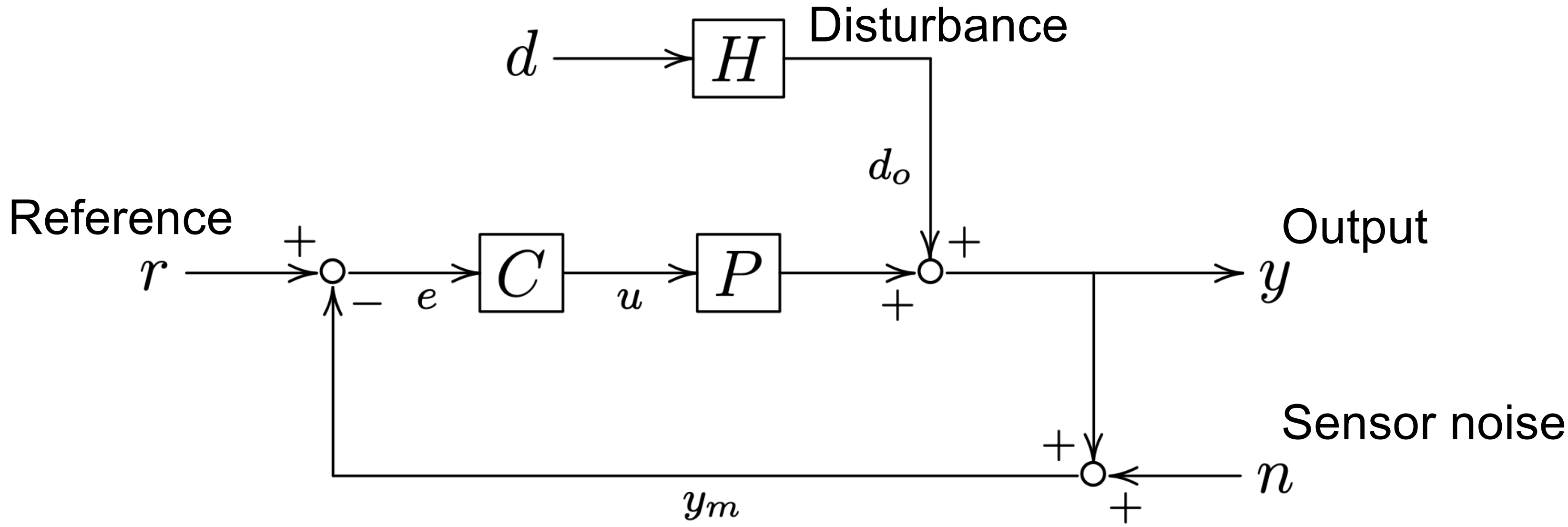
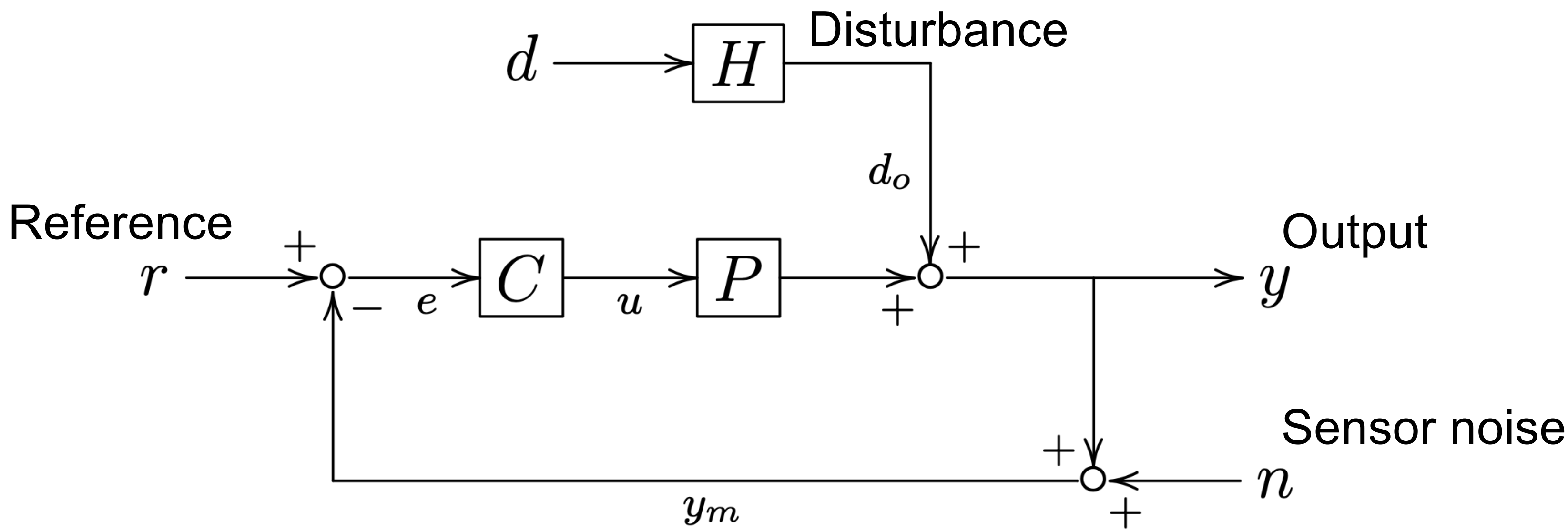


$$\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} G_{r\to y} & G_{d_o\to y} & G_{n\to y} \\ G_{r\to u} & G_{d_o\to u} & G_{n\to u} \end{bmatrix} \begin{bmatrix} r \\ d_o \\ n \end{bmatrix} = \begin{bmatrix} \frac{PC}{1+PC} & \frac{1}{1+PC} & \frac{-PC}{1+PC} \\ \frac{C}{1+PC} & \frac{-C}{1+PC} & \frac{-C}{1+PC} \end{bmatrix} \begin{bmatrix} r \\ d_0 \\ n \end{bmatrix}$$

$$y = \begin{bmatrix} \frac{PC}{1+PC} & \frac{1}{1+PC} & \frac{-PC}{1+PC} \end{bmatrix} \begin{bmatrix} r \\ d_0 \\ n \end{bmatrix}$$

Arithmetic of feedback loops



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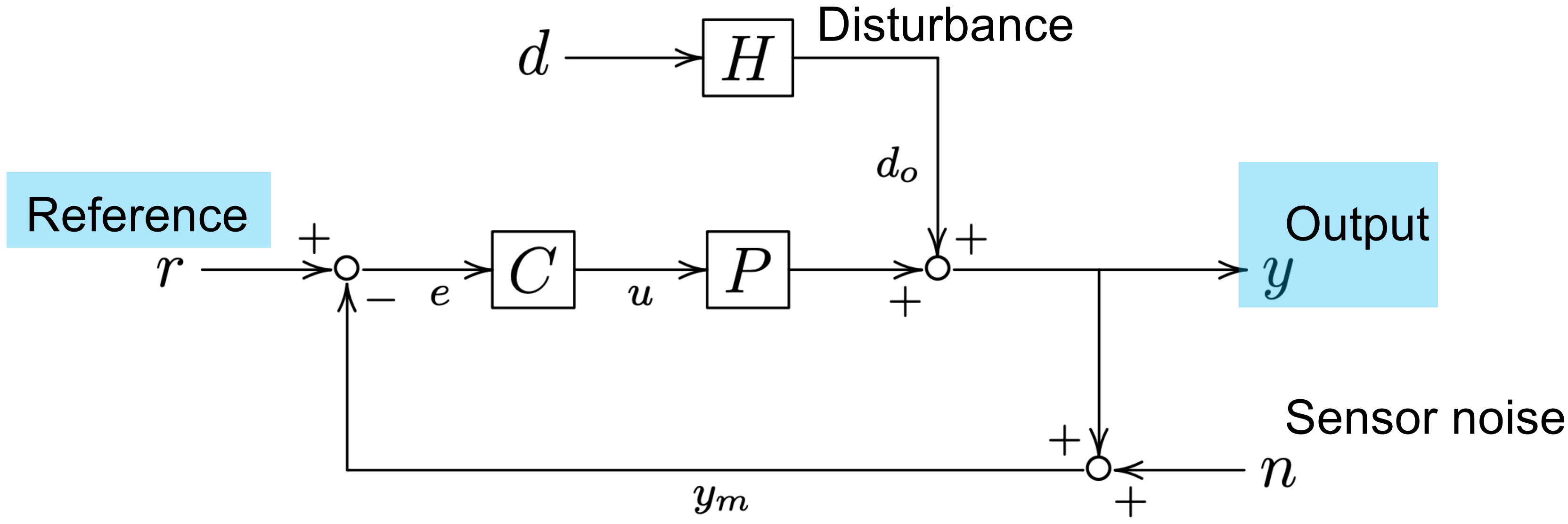
#2

Goals of feedback

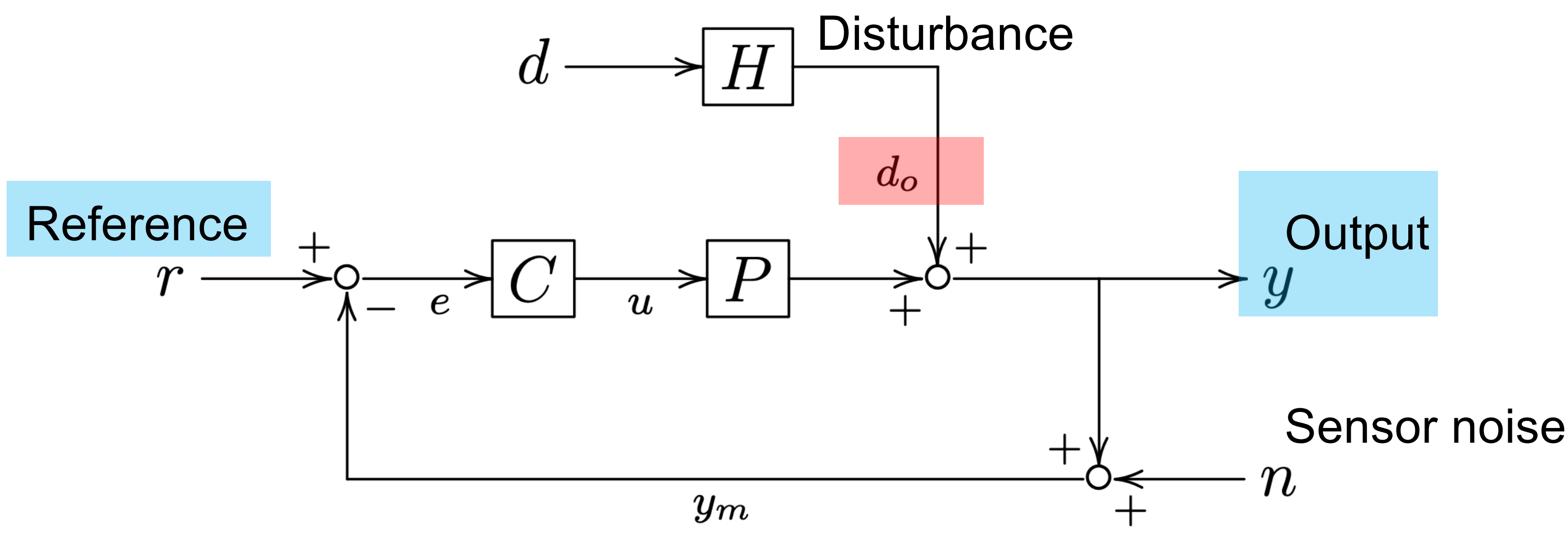


$$y = \begin{bmatrix} \frac{PC}{1+PC} & \frac{1}{1+PC} & \frac{-PC}{1+PC} \end{bmatrix} \begin{bmatrix} r \\ d_0 \\ n \end{bmatrix}$$

Desired: ~1

Complementary Sensitivity Function

Goals of feedback

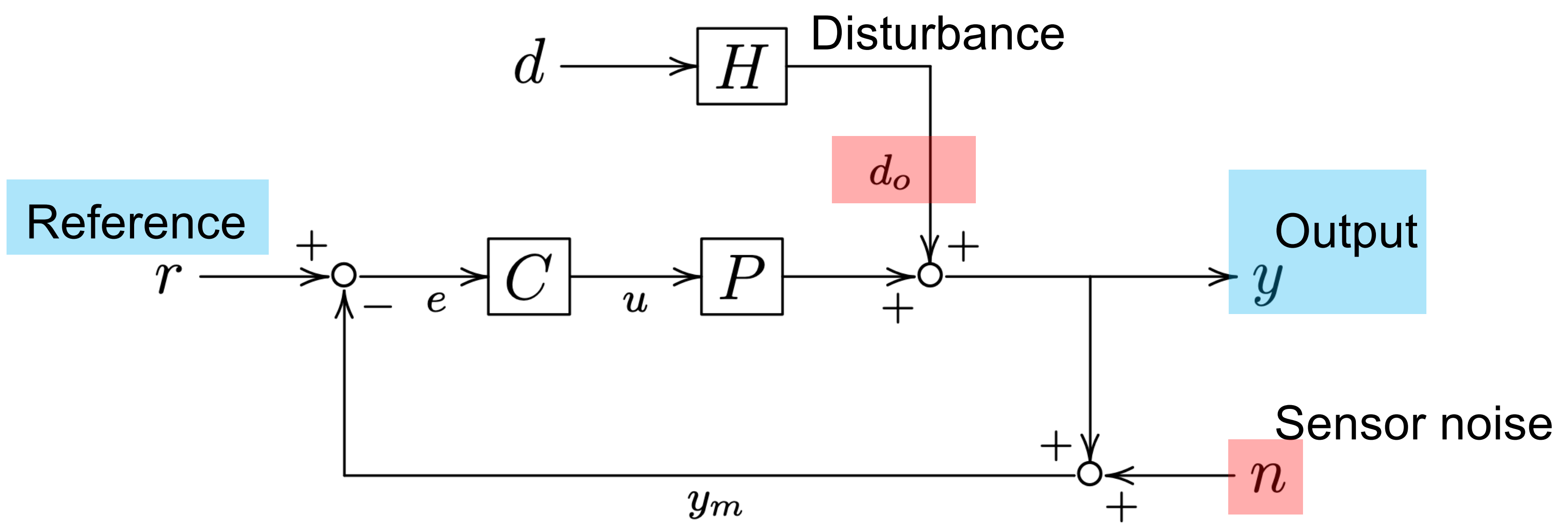


$$y = \begin{bmatrix} \frac{PC}{1+PC} & \frac{1}{1+PC} & \frac{-PC}{1+PC} \end{bmatrix} \begin{bmatrix} r \\ d_0 \\ n \end{bmatrix}$$

Desired: ~1 ~0

Complementary Sensitivity Function

Goals of feedback



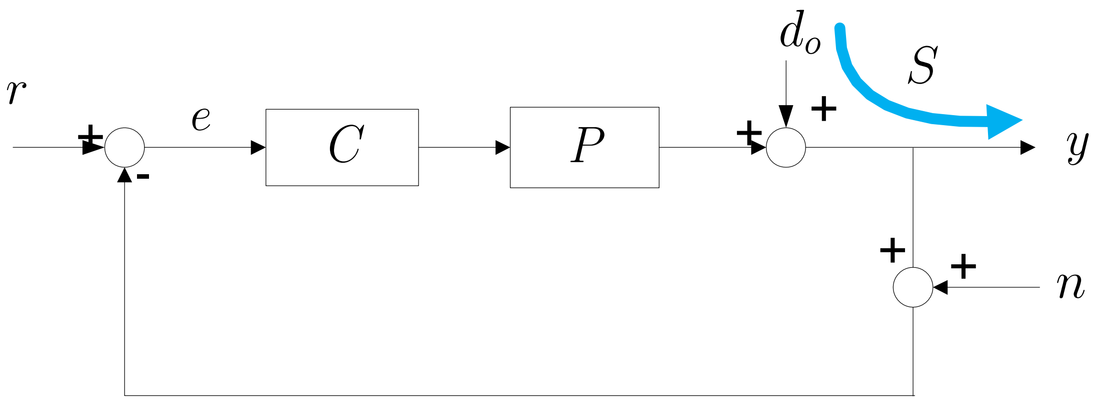
$$y = \left[\frac{PC}{1+PC} \quad \frac{1}{1+PC} \quad \frac{-PC}{1+PC} \right] \begin{bmatrix} r \\ d_0 \\ n \end{bmatrix} \begin{matrix} \text{Reference} \\ \text{Disturbance} \\ \text{Sensor noise} \end{matrix}$$

Desired: ~1   ~0   ~0

Can't do well on both!

#3

Tradeoffs



$$y = \begin{bmatrix} \frac{PC}{1+PC} & \frac{1}{1+PC} & \frac{-PC}{1+PC} \end{bmatrix} \begin{bmatrix} r \\ d_0 \\ n \end{bmatrix}$$

Sensitivity Function: $S = (1 + PC)^{-1}$

Complementary Sensitivity Function:

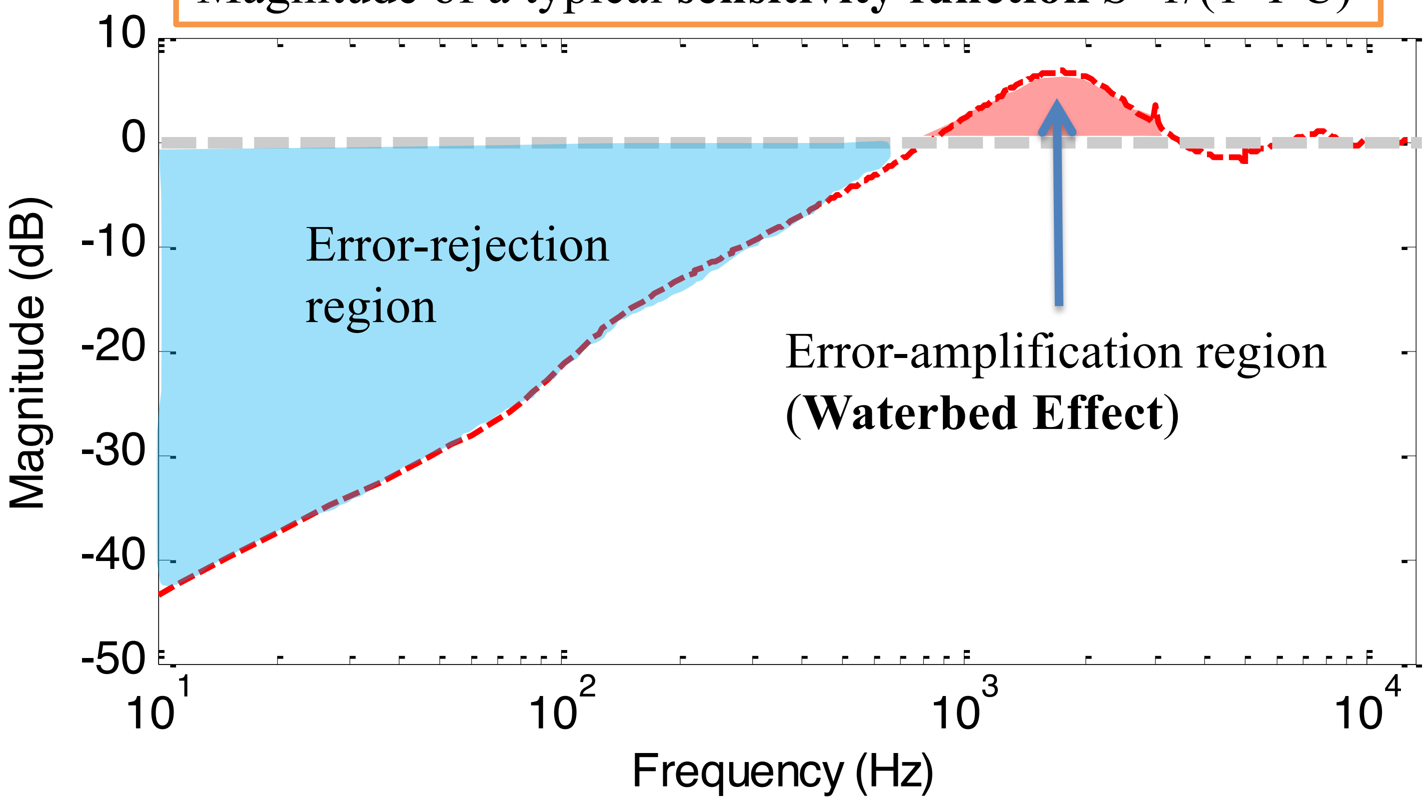
$$T = PC(1 + PC)^{-1}$$

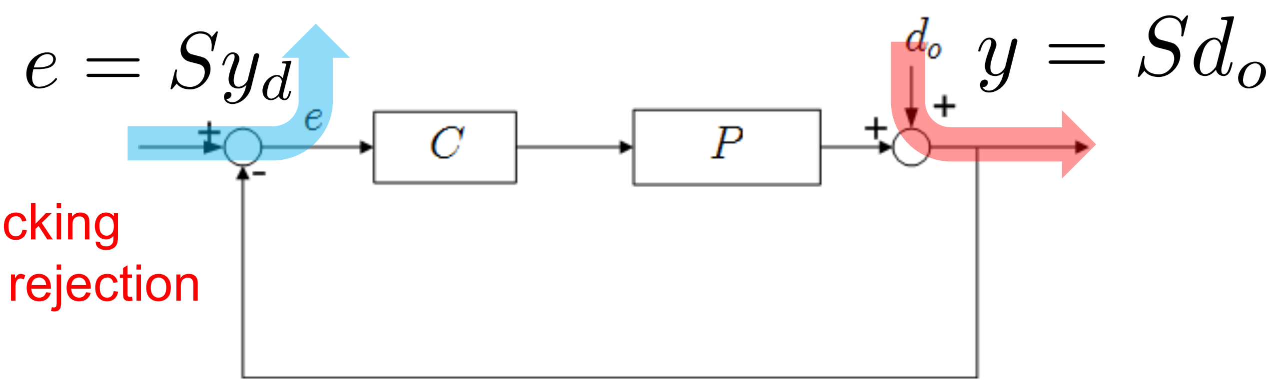
Fundamental Constraint: $S + T = 1$

#4

Loop shaping

Magnitude of a typical **sensitivity function** S=1/(1+PC)

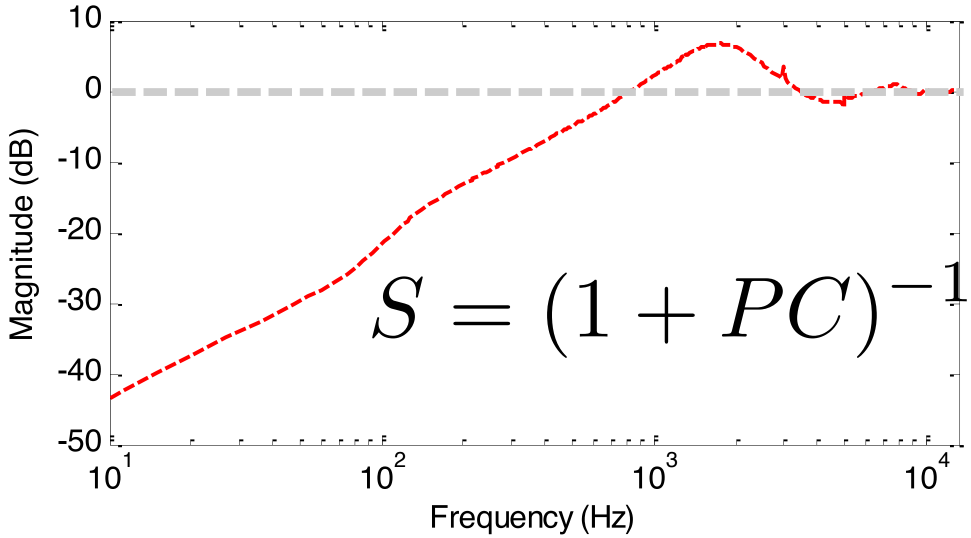




S defines the tracking and disturbance rejection performances

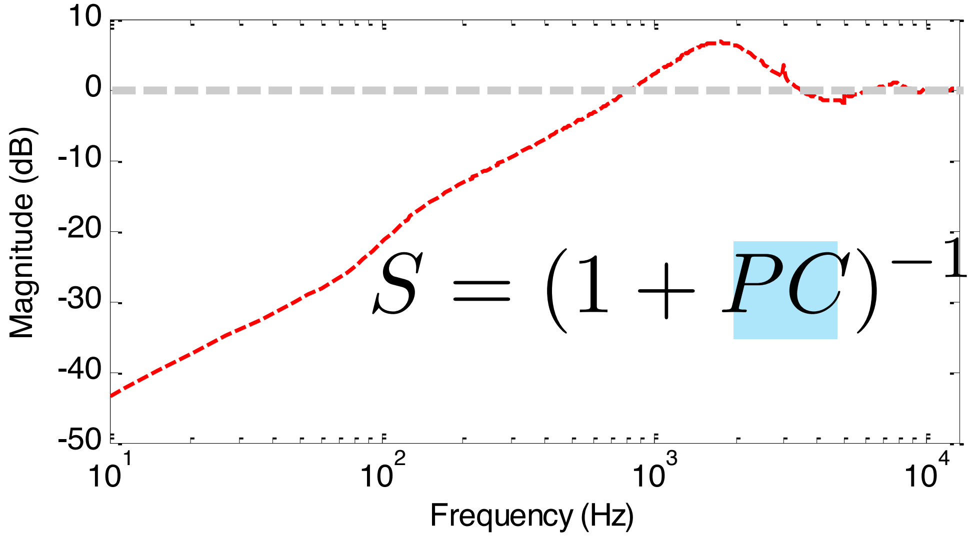
#5

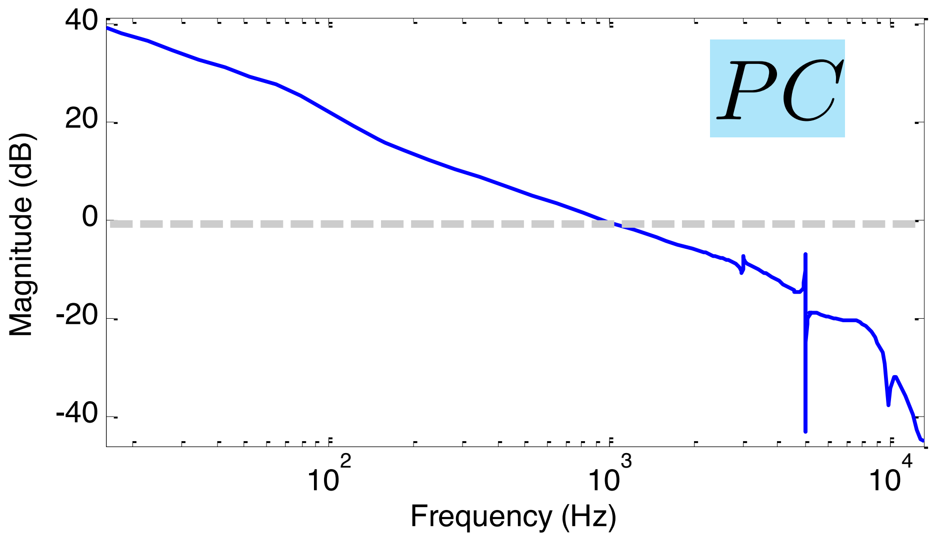
High-gain feedback



#5

High-gain feedback





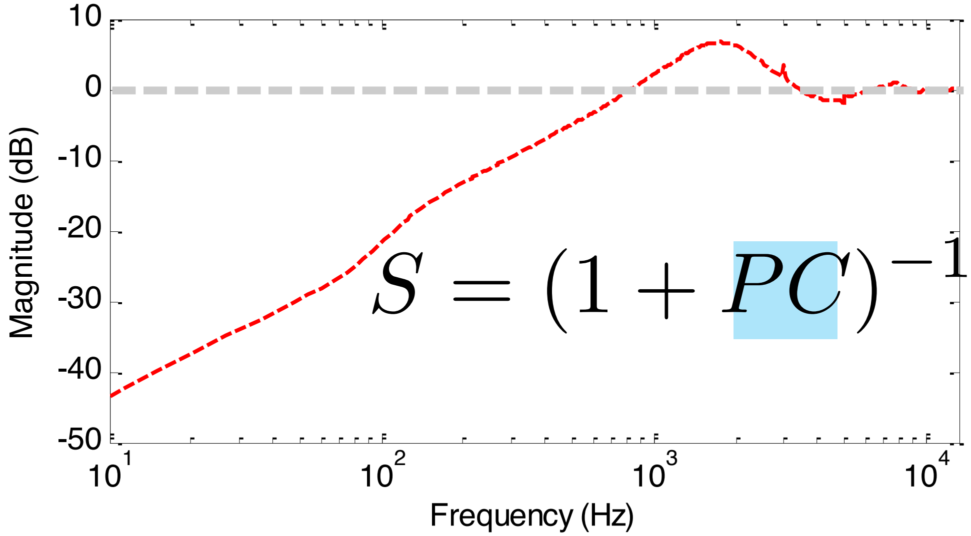
small gain in ***S***

⇔

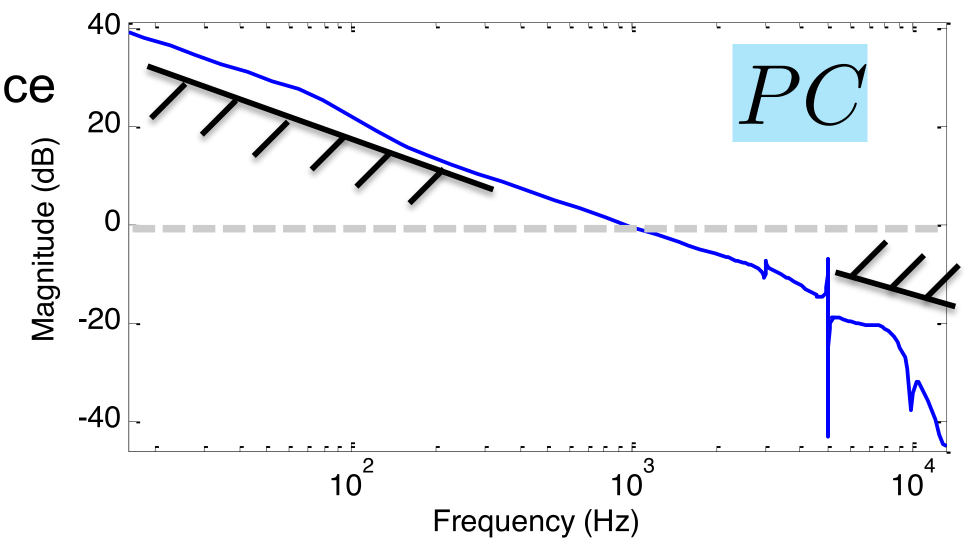
high gain in ***PC***

#5

High-gain feedback



Typical high-gain control for performance at low frequency

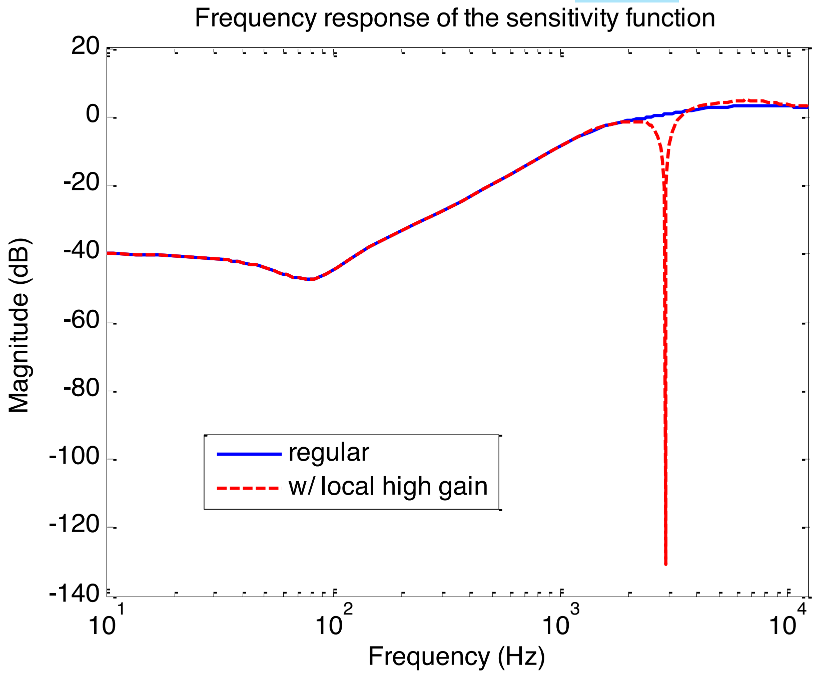


Typical low-gain control for robustness at high frequency

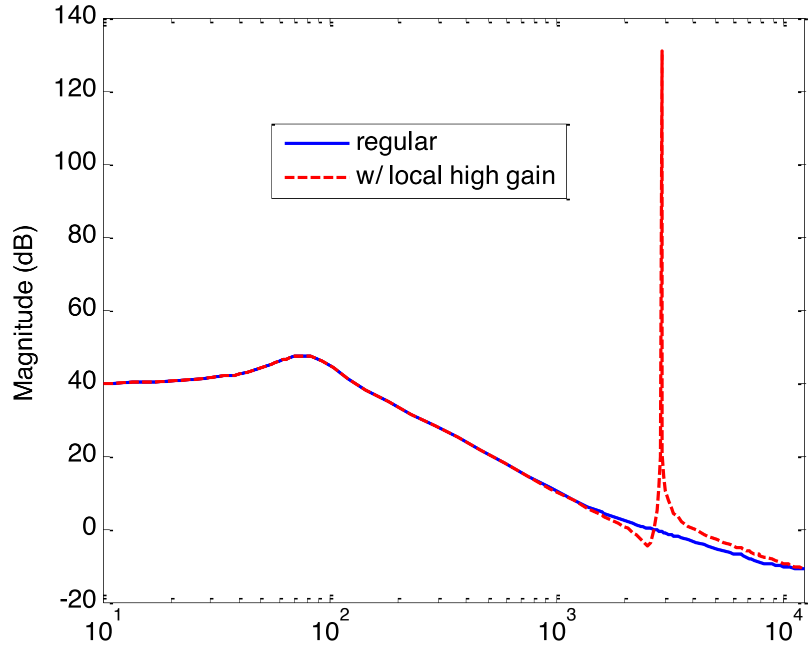
#5

Local high-gain feedback

$$S = (1 + PC)^{-1}$$



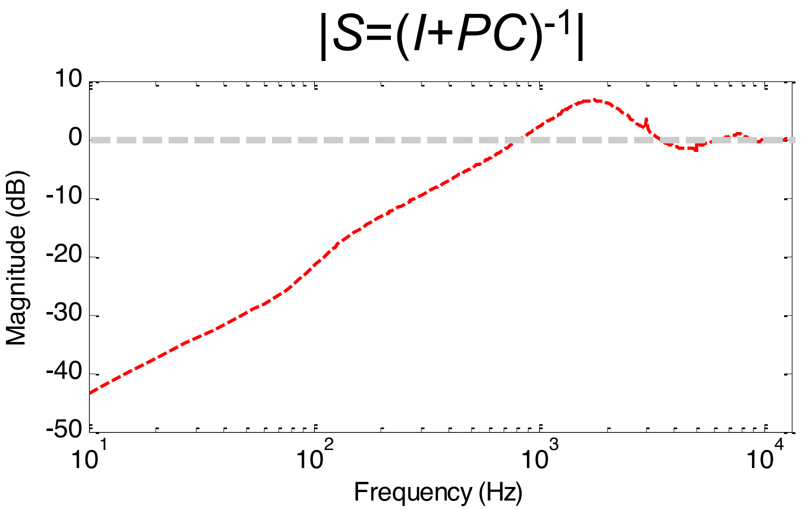
$$PC$$



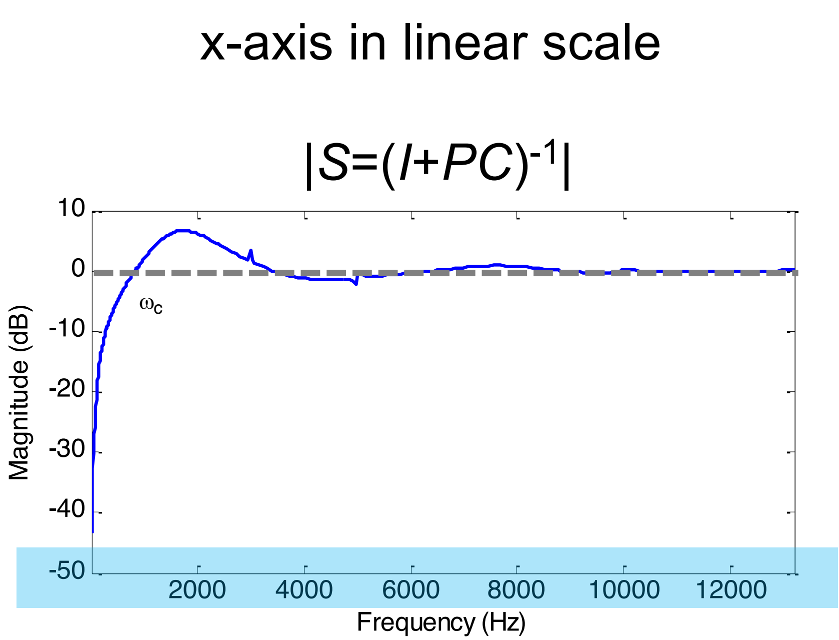
#6

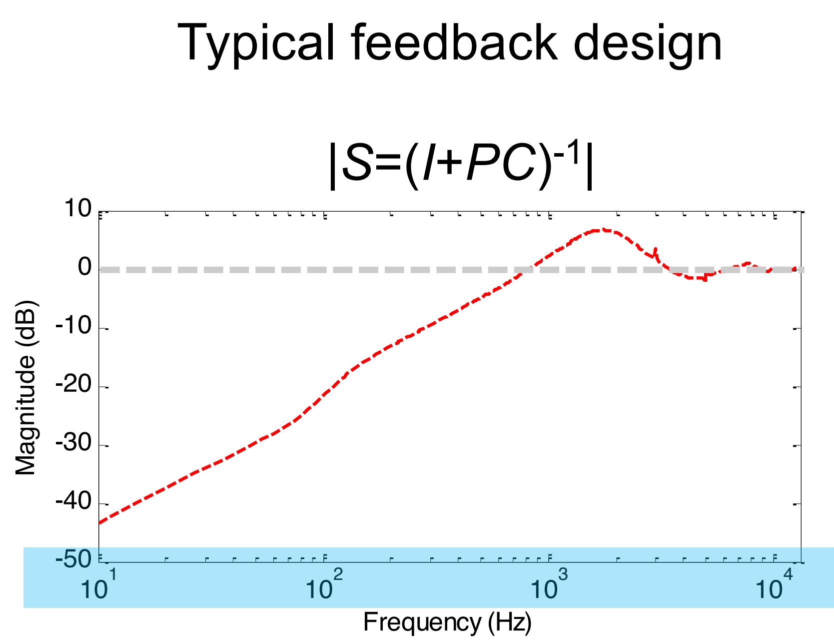
Bode's Integral

Typical feedback design



Bode's Integral

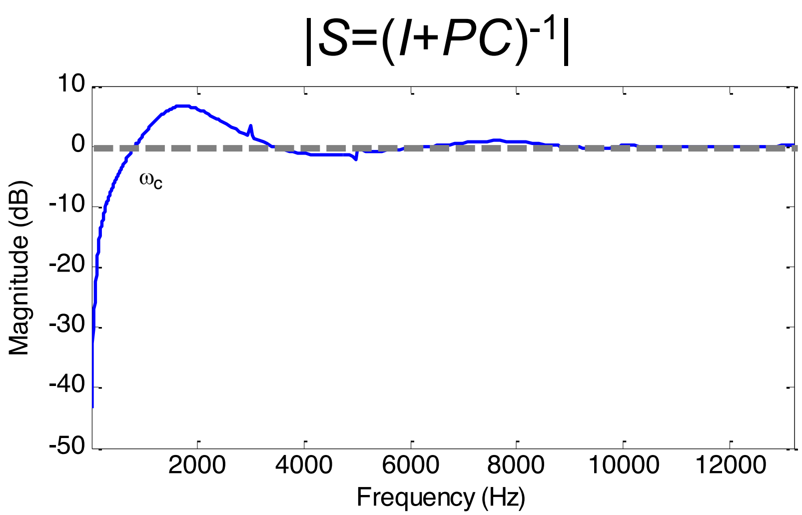




#6

Bode's Integral

Theorem. Let $S(s) = 1/(1+L(s))$. If $L(s)$ and $S(s)$ are both rational and stable. Then



$$\frac{1}{\pi}\int_0^\infty \ln|S(j\omega)|d\omega = \frac{-1}{2}k_s$$

$$k_s = \lim_{s\to\infty} sL(s)$$

Bode's Integral

Theorem. Let $S(s) = 1/(1+L(s))$. If $L(s)$ and $S(s)$ are both rational and stable. Then

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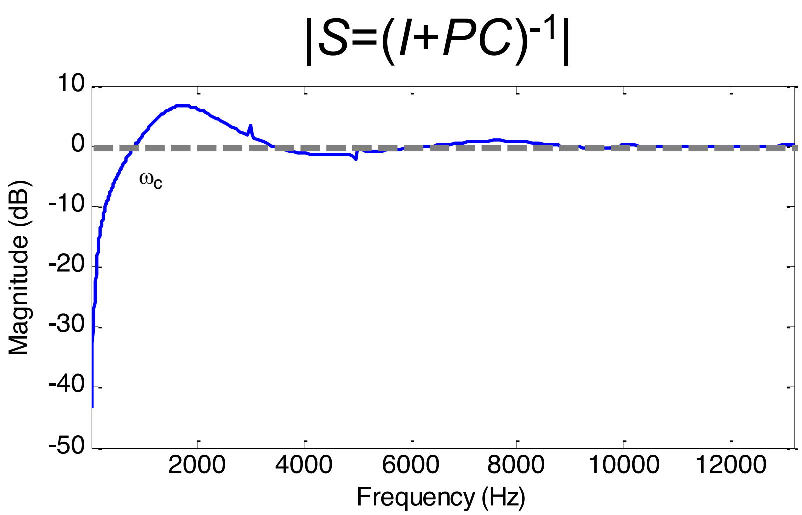


Special Case: If the relative degree of $L(s)$ is larger than or equal to 2, then

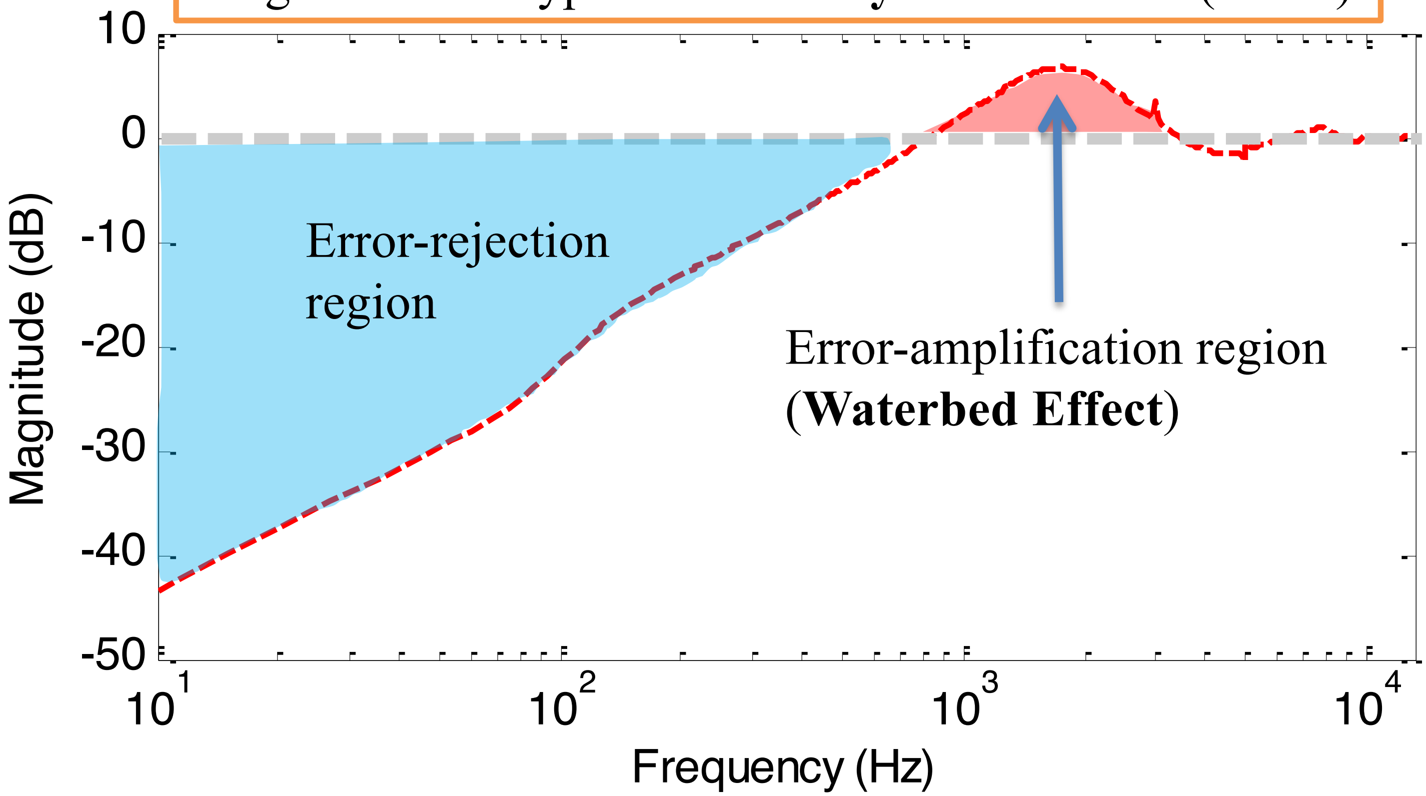
$$\frac{1}{\pi}\int_0^\infty \ln|S(j\omega)|d\omega = 0$$

#6

Bandwidth limitation

Recall:

Magnitude of a typical **sensitivity function** S=1/(1+PC)



Bode's Integral:

$$\frac{1}{\pi}\int_0^\infty \ln|S(j\omega)|d\omega = 0$$

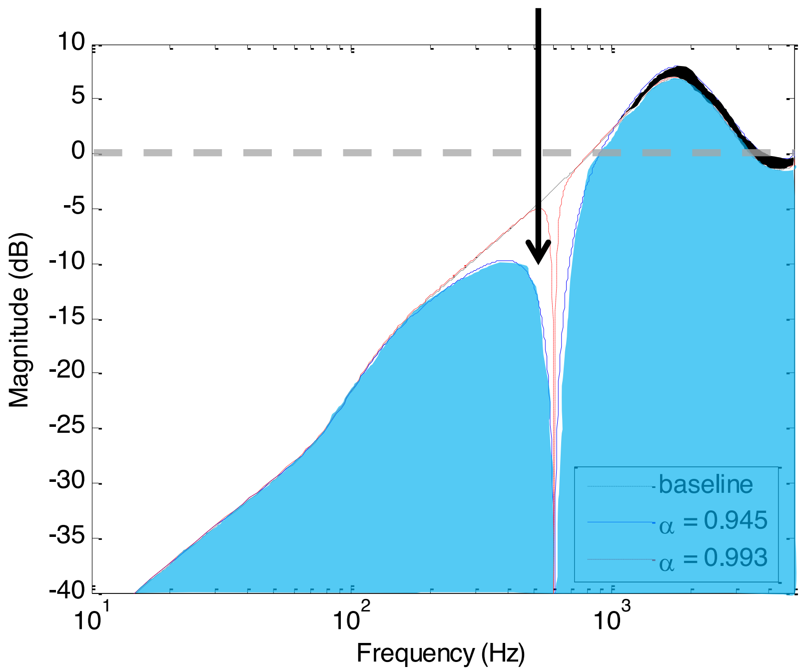
Hence it is inevitable to have the error-amplification region.

Waterbed effect: pushing down ***S*** in one region causes amplification in some other region.

#6

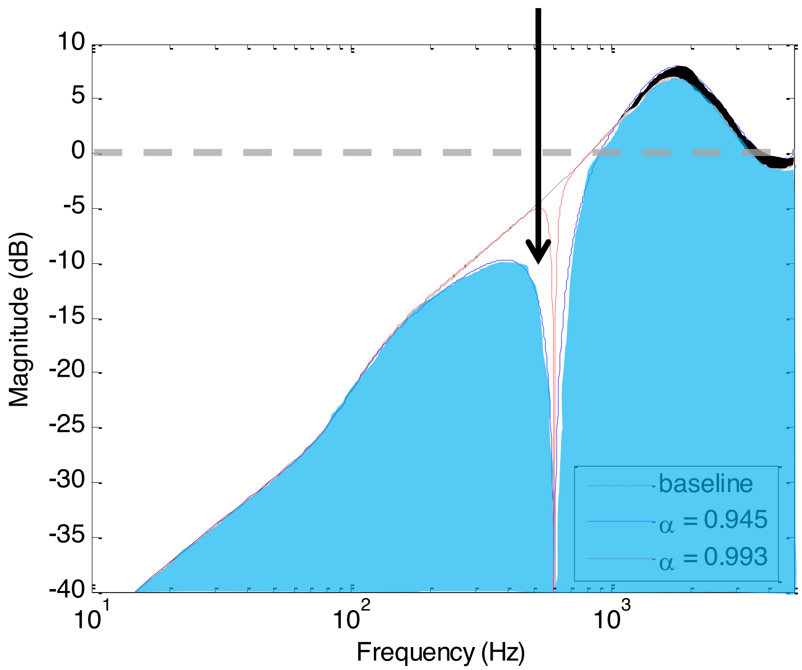
Waterbed Effect

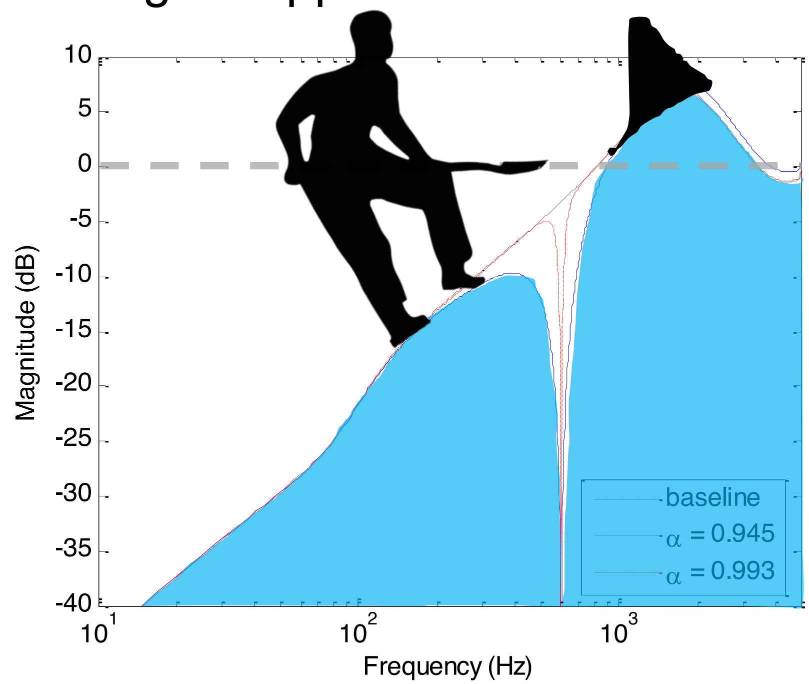
So to achieve this,



Waterbed Effect

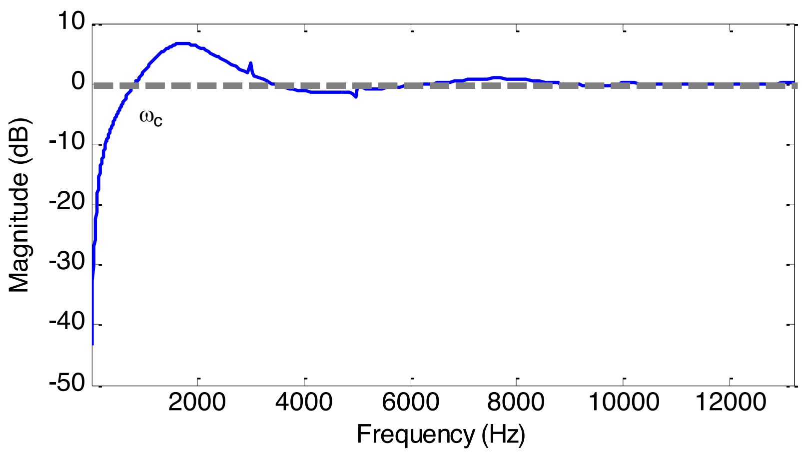
So to achieve this, this might happen…





#6

General Bode's Integral



Theorem. Let *S*(*s*) = 1/(1+*L*(*s*)). If *L*(*s*) and *S*(*s*) are both rational, *S*(*s*) is stable, and *L*(*s*) has unstable or imaginary unstable poles $\{p_k\}_{k=1}^{q}$. Then

$$\frac{1}{\pi}\int_0^\infty \ln\left|S\left(j\omega\right)\right|d\omega = \sum_{k=1}^{q} p_k$$

Proof: complex analysis, analytic functions, Cauchy Integral.

#7

Limitations from unstable zeros

- Example: $P = sP_{else}$ ➔ constant inputs can't impact the output

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- More consequences:
  - S *always* has magnitudes larger than one

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Limitations from unstable zeros

- Example: $P = sP_{else}$ ➔ constant inputs can't impact the output
- More consequences:
  - S *always* has magnitudes larger than one:

$$P(\sigma_o) = 0 \qquad S(\sigma_o) = 1/\left(1 + 0 \times C(\sigma_o)\right) = 1$$

Closed-loop stability ➔ S is analytic in the right-half plane

maximum modulus theorem ➔

$$\max_{\omega} |S(j\omega)| \geq |S(\sigma_o)| = 1$$

#7

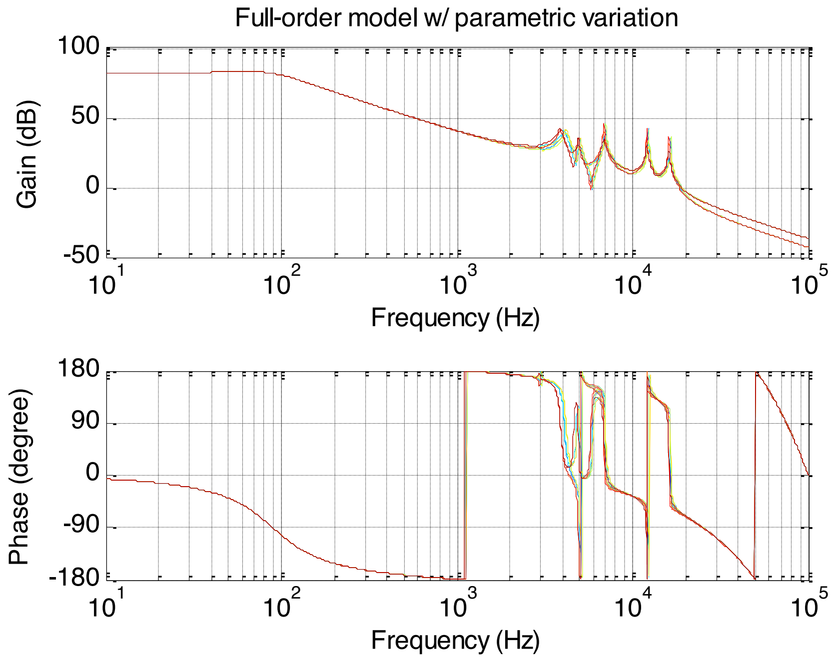
Limitations from unstable zeros

- Example: $P = sP_{else}$ ➔ constant inputs can't impact the output
- More consequences:
  - S *always* has magnitudes larger than one
  - Not able to perform accurate system ID
  - High-gain instability
  - Step responses can have initial undershoot
  - etc

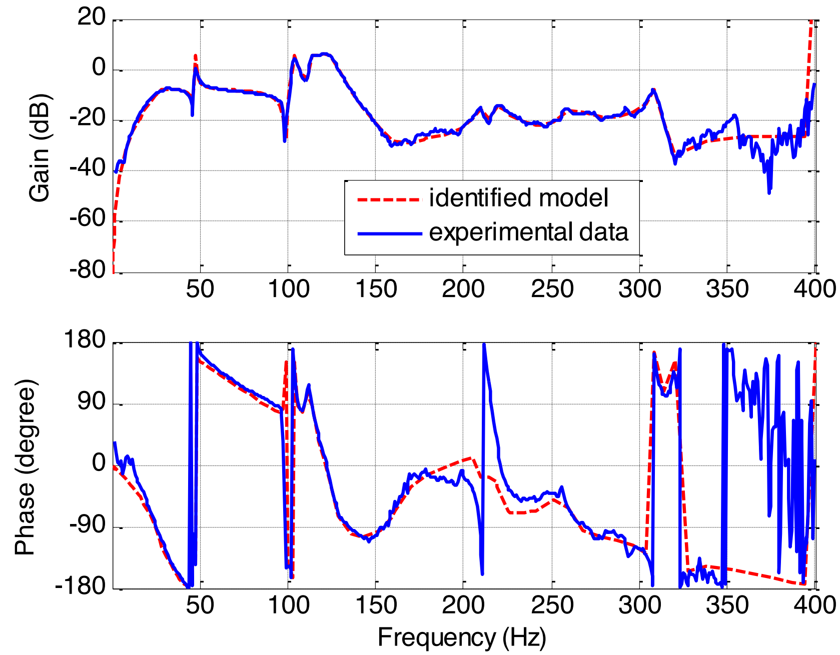
Resonance and anti-resonance

- Typical in mechanical systems.
- Usually identified experimentally.

HDD

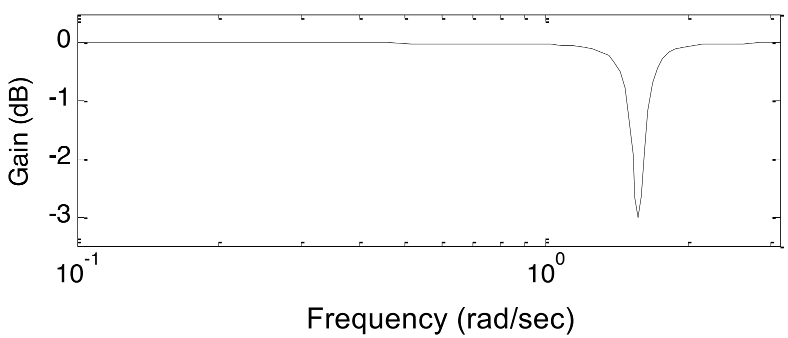


Active suspension



#8

Notch filters

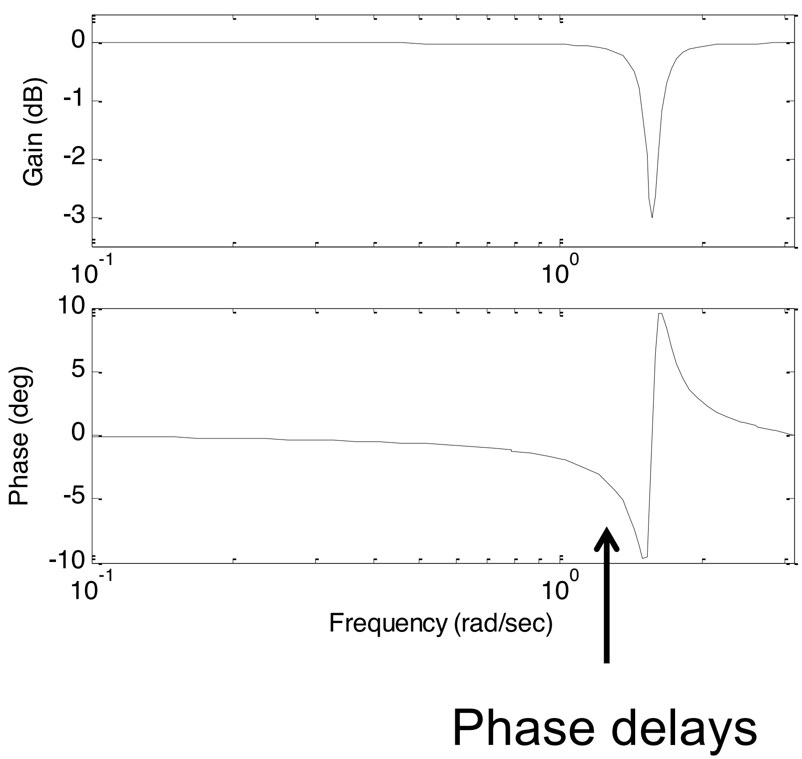


Notch filtering: one common technique to handle resonances

Fundamental constraint in notch filtering: introduces phase delays to the system

#8

Magnitude-phase relationship



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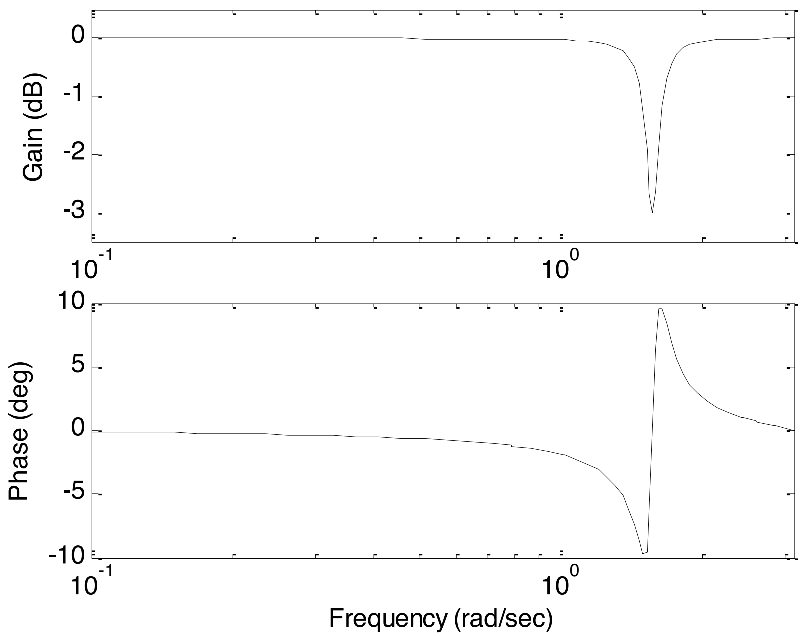

Theorem (Bode's Phase Formula). *If L is a minimum-phase continuous-time transfer function, then its phase is uniquely defined by its gain, according to*

$$\angle L(j\omega) = \int_{-\infty}^{\infty} \frac{d \ln |L(e^{\nu}\omega)|}{d\nu} \psi(\nu)\, d\nu$$

where

$$\psi(\nu) = \frac{1}{\pi} \ln \frac{e^{|\nu|/2} + e^{-|\nu|/2}}{e^{|\nu|/2} - e^{-|\nu|/2}}.$$

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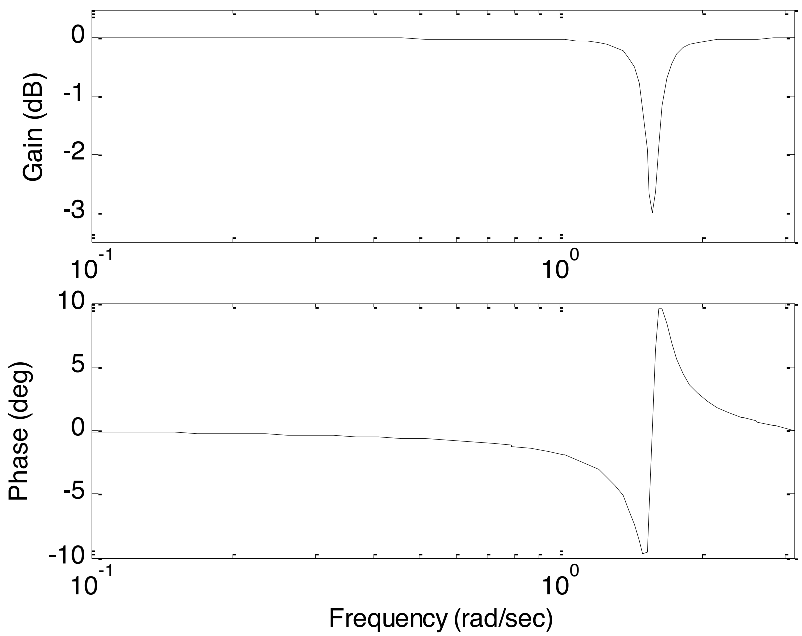
Magnitude-phase relationship



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Slope of magnitude response

where

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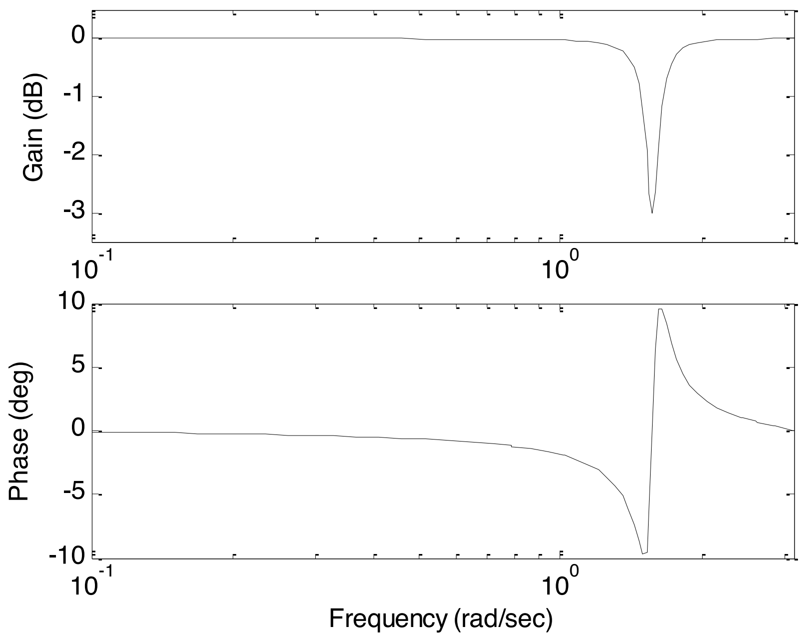
Magnitude-phase relationship



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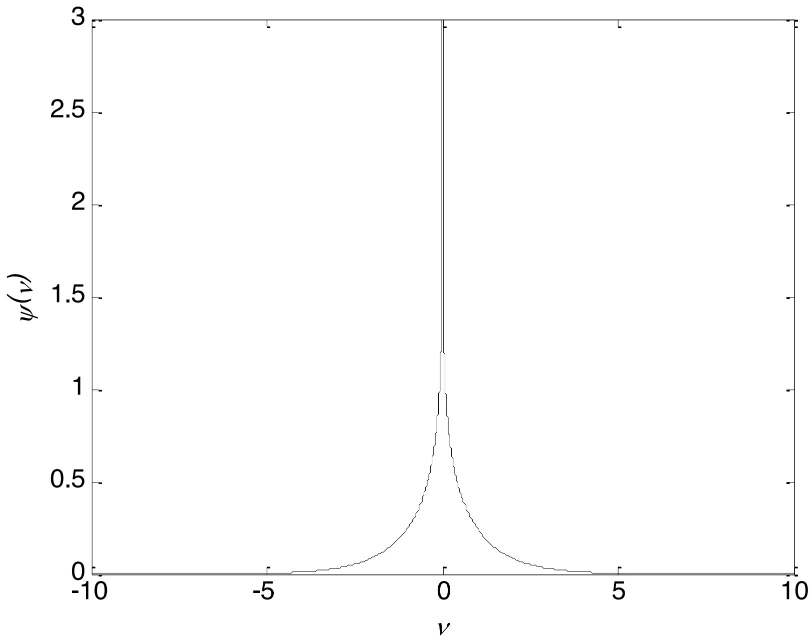
Approximately an impulse at 0

#8

Magnitude-phase relationship



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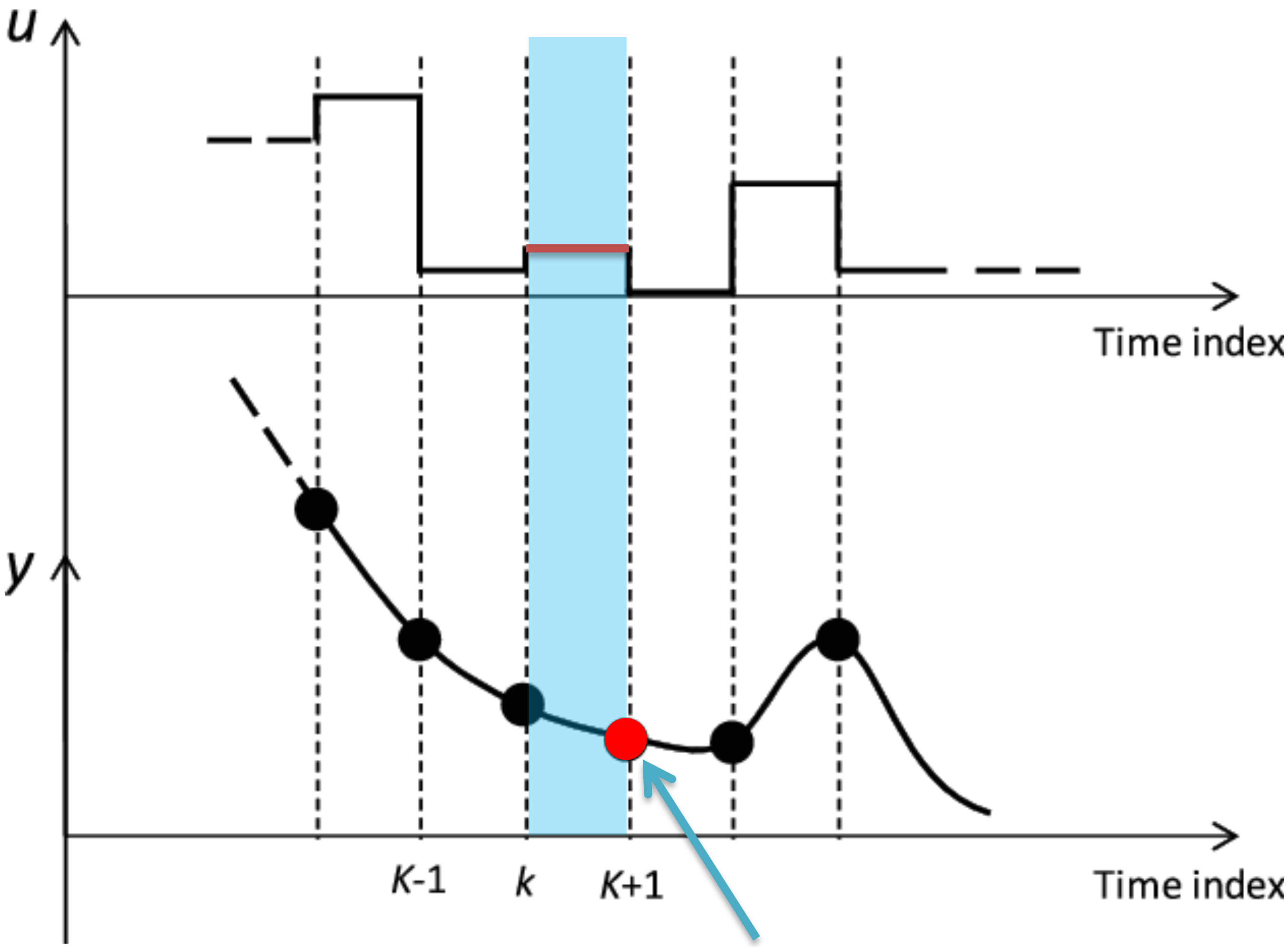
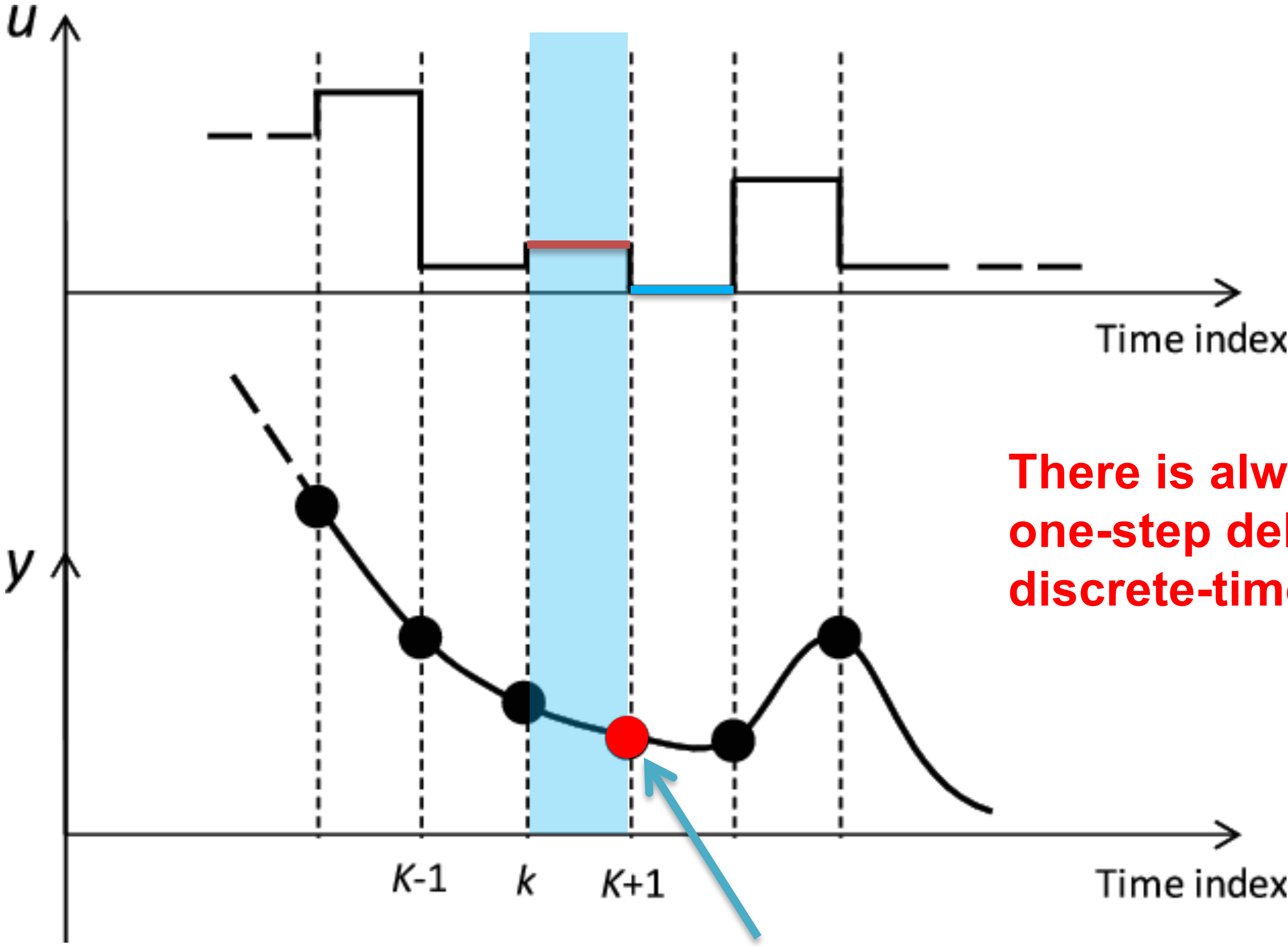
Discrete-time plant delay



$$y(k+1) = f(u(k), u(k-1), \ldots)$$

Discrete-time plant delay

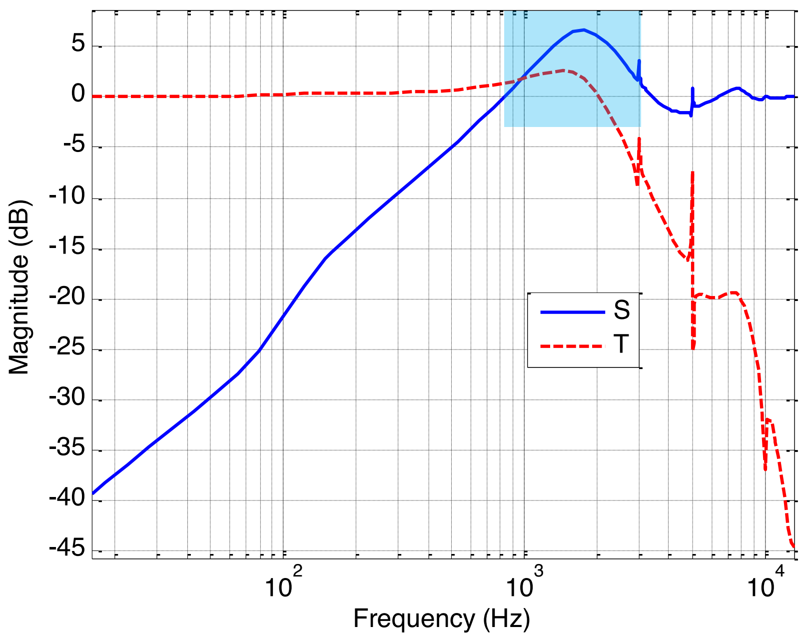


There is always at least one-step delay in discrete-time systems!

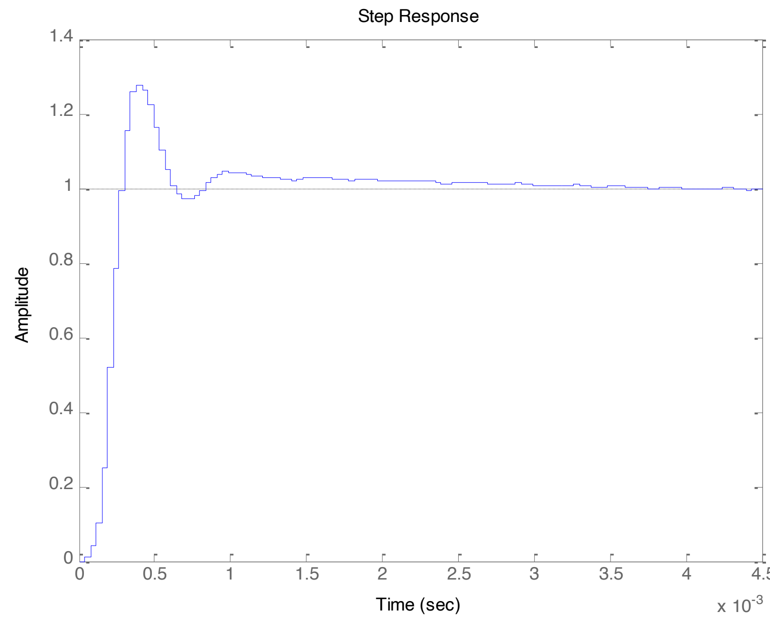
$$y(k+1) = f(u(k), u(k-1), \ldots)$$

#10 Bandwidth and rise time: practical application

Actual system frequency response

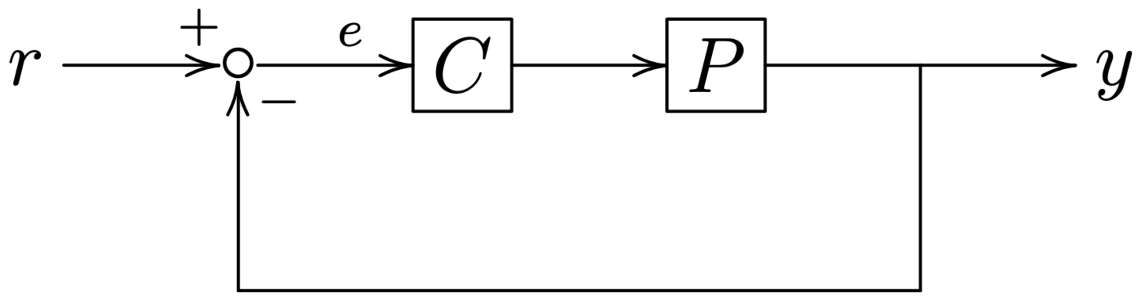


Step response of a high-order closed-loop system



#11

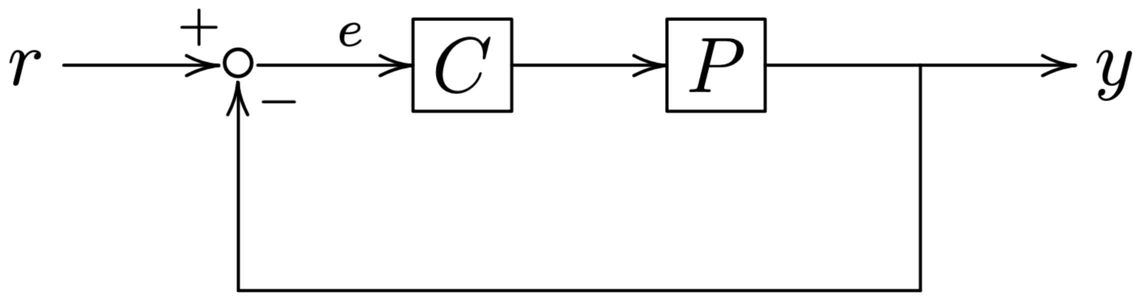
Sampling-time selection



- Rule of thumb:
  - Sampling frequency $\approx$ 10 ~ 20 bandwidth (in Hz)

#11

Sampling-time selection



- Rule of thumb:
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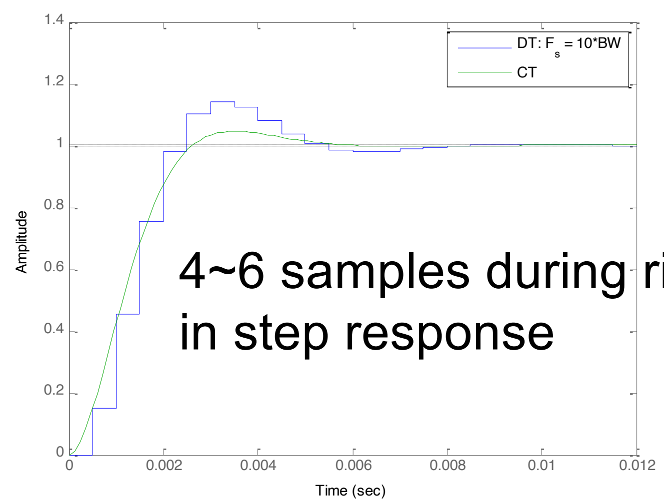
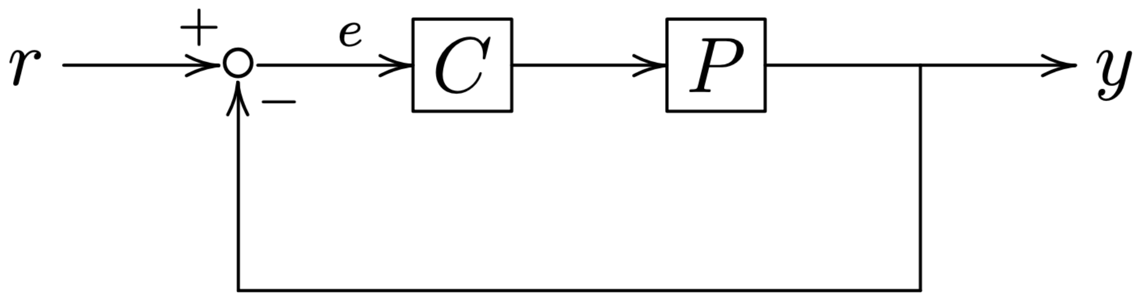


4~6 samples during rise time in step response

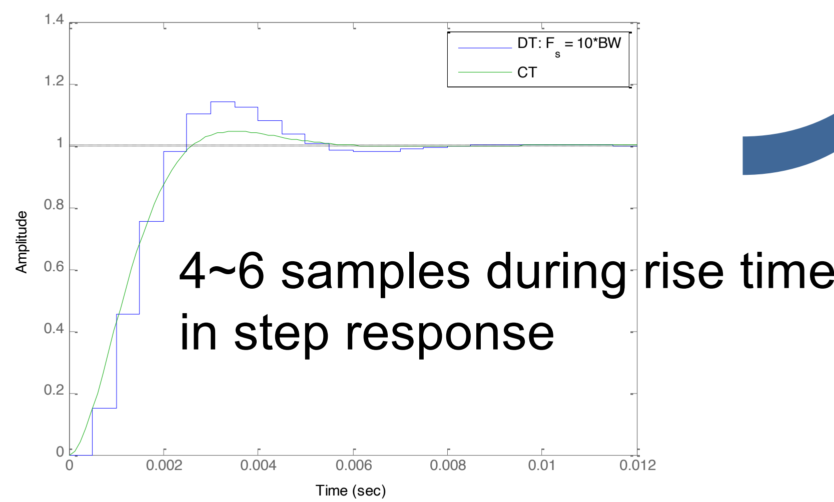
#11

Sampling-time selection

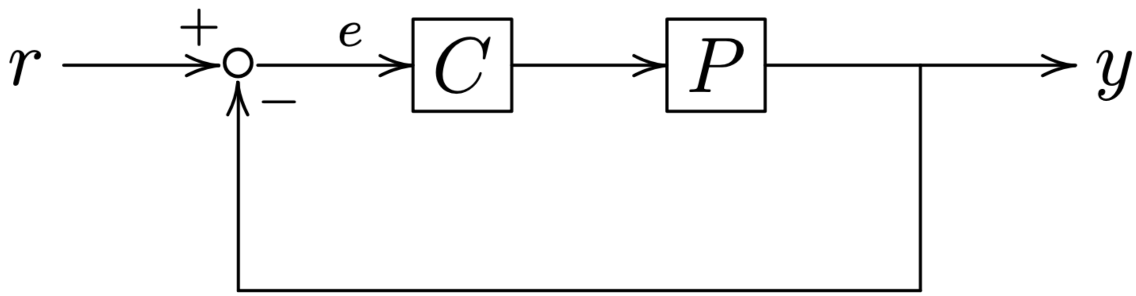


Intuition: 20 = the number of letters in “sampling frequencies”

- Rule of thumb:
  - Sampling frequency $\approx$ 10 ~ 20 bandwidth (in Hz)



#11 Sampling-time selection: example



Example:

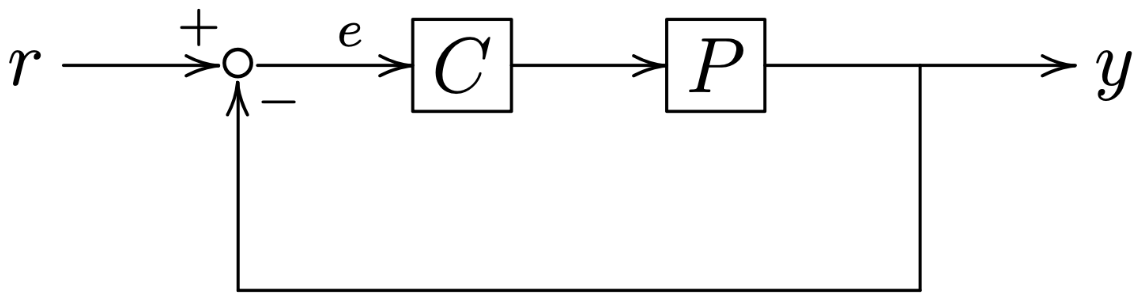
$$P = k$$

$$C = \frac{\omega_n^2}{s^2 + 2\omega_n s}\frac{1}{k}$$

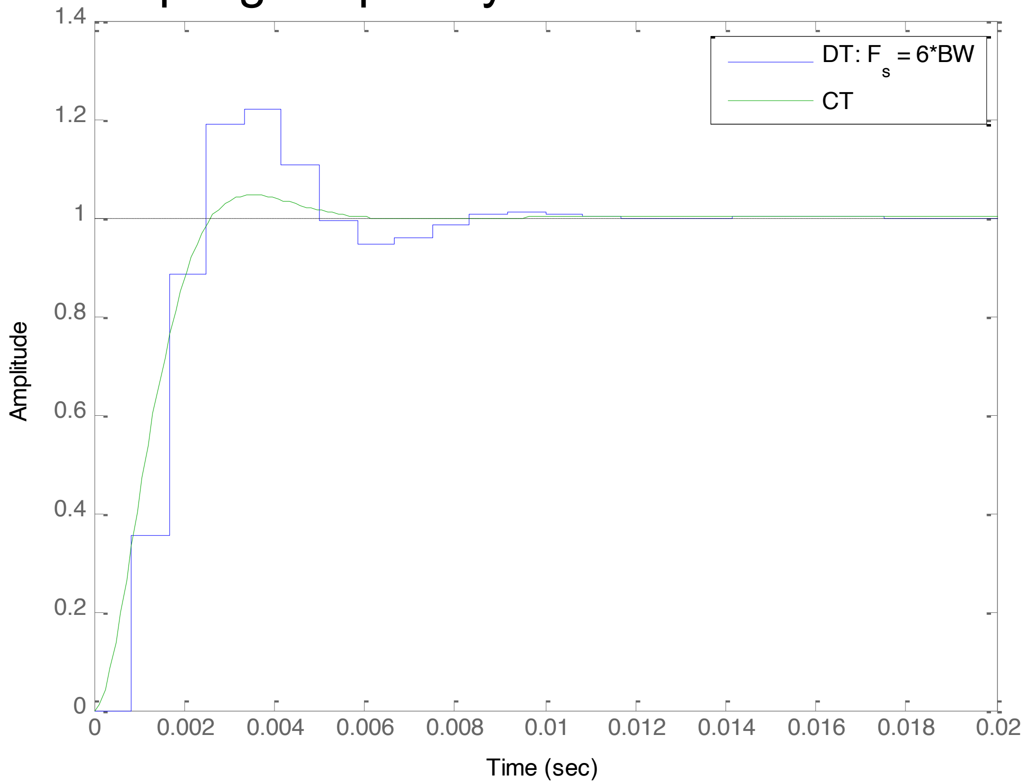
$$S = \frac{s^2 + 2\omega_n s}{s^2 + 2\omega_n s + \omega_n^2}$$

$$T = \frac{\omega_n^2}{s^2 + 2\omega_n s + \omega_n^2}$$

#11 Sampling-time selection: example



Sampling frequency = 6 x bandwidth



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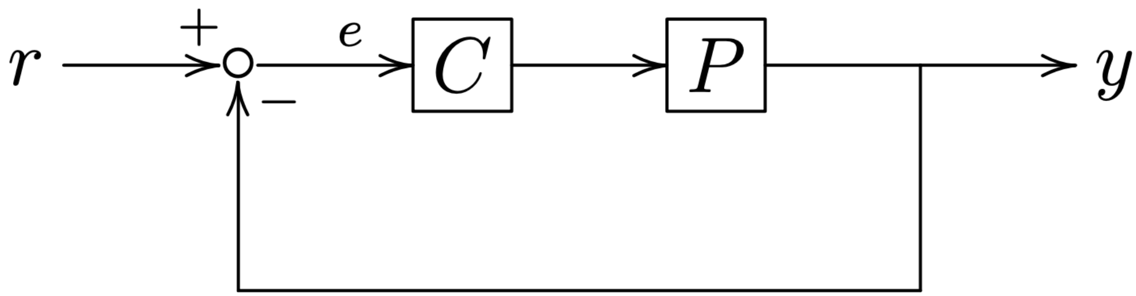
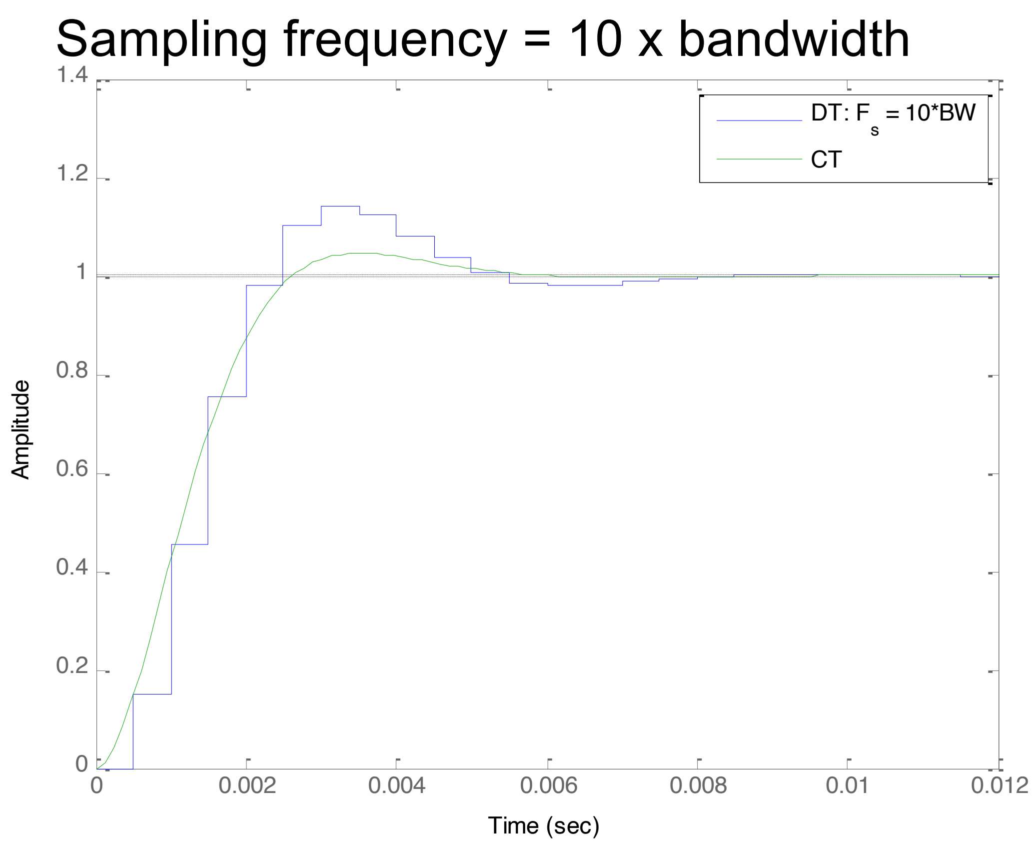
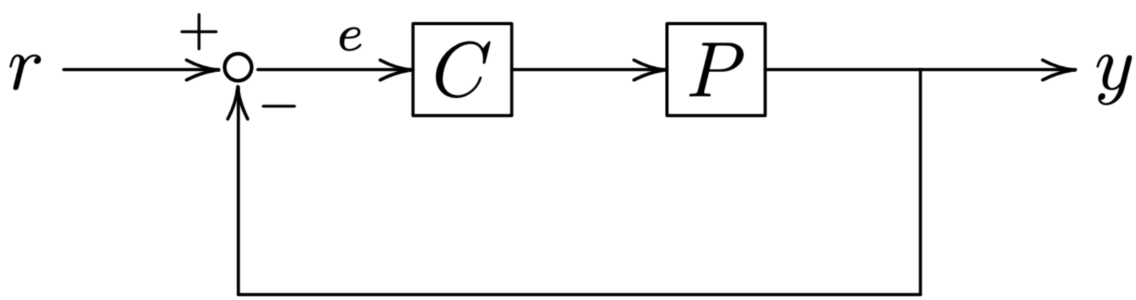
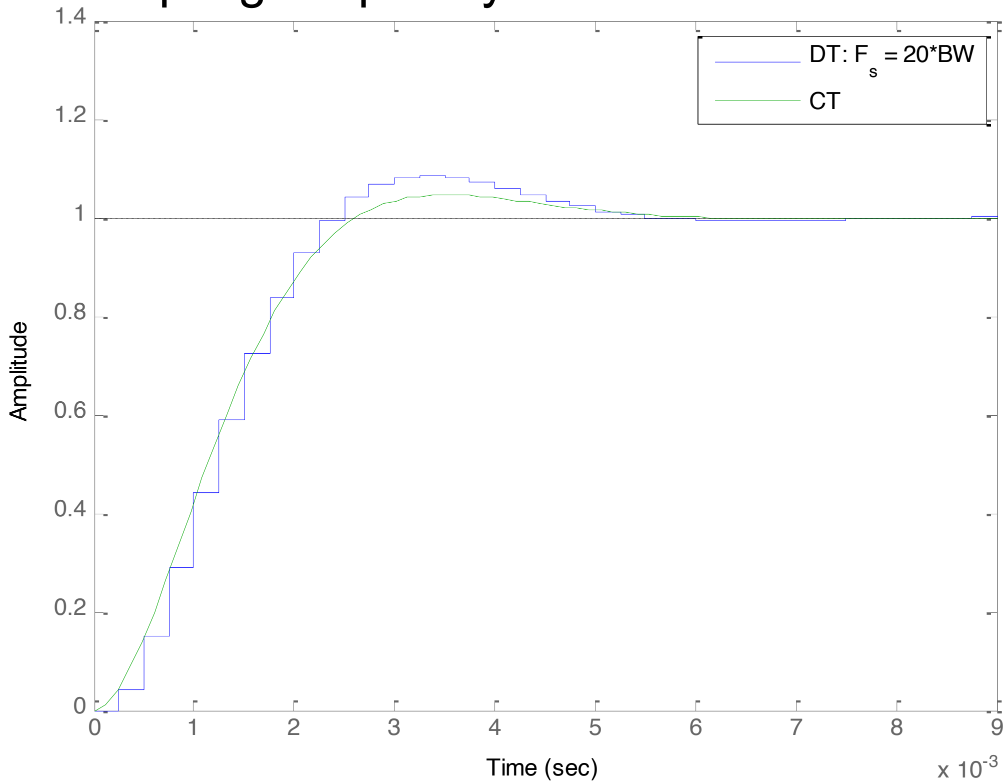




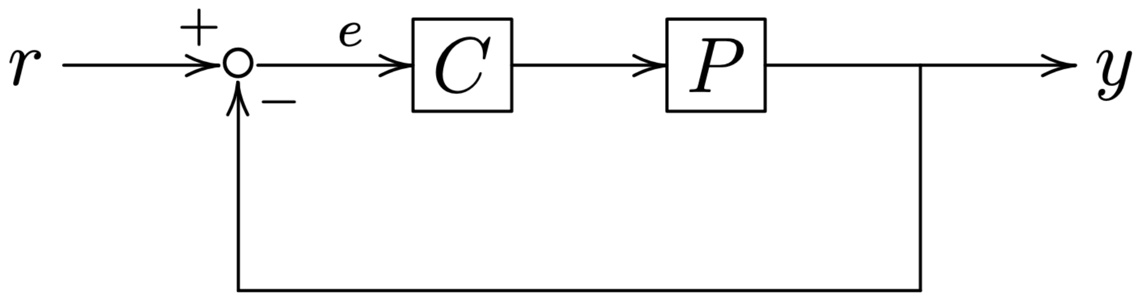
#11 Sampling-time selection: example



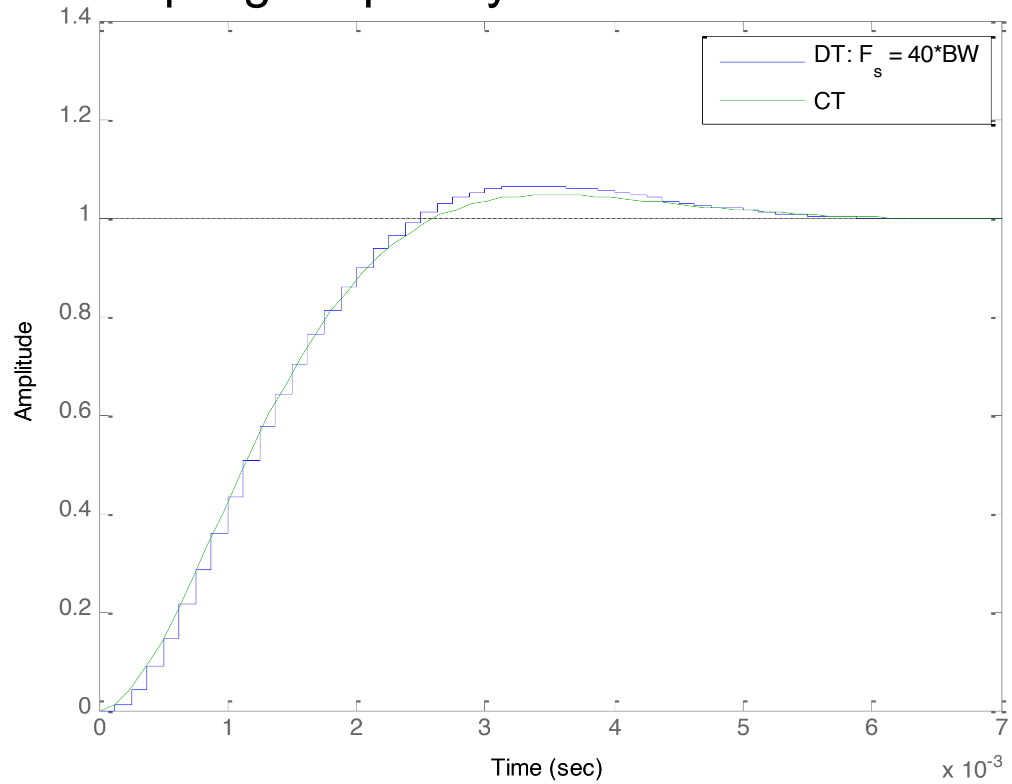
Sampling frequency = 20 x bandwidth



#11 Sampling-time selection: example



Sampling frequency = 40 x bandwidth



Related active research field

- Flexible loop shaping
- Vibration rejection and motion control
- MIMO loop shaping
- Delay compensation
- Adaptive control
- Nonlinear control and breaking the waterbed effect